RADIATION FROM AN OSCILLATING MAGNETIC DIPOLE IN A STREAMING PLASMA*

by

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A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Physics and Astronomy in the Graduate College of the University of Iowa

June 1967

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* Research supported in part by the National Aeronautics and Space Administration under Contracts NGR-16-001-043 and NsG-233-62

** Part of this work was carried out when the author held a N.A.S.A. Graduate Research Fellowship

ACKNOWLEDGMENTS

I particularly wish to thank Dr. D. C. Montgomery for his guidance and advice during the course of this study. I am also indebted to Dr. D. A. Gurnett and Dr. G. R. Joyce for their fruitful suggestions and to Dr. J. A. Van Allen for his encouragement and support. To Miss Ann Schmeichel, who typed the manuscript, and to Mr. David Lunde, who drafted the figures, I also express my appreciation.

ABSTRACT

The electromagnetic field of an oscillating magnetic dipole is calculated, assuming that the dipole is immersed in a cold streaming plasma. The amplitude of the magnetic dipole moment, assumed known, is taken to be sufficiently weak that the linearized cold plasma equations may be used to describe the response of the plasma.

The resulting field of the dipole is rather different from the field that would result if the plasma were not streaming. In particular, a longitudinal electrostatic field appears as a consequence of the plasma's motion. The far field of the dipole is such that the Poynting vector is not purely radial, but is tilted against the direction of the zeroth order plasma flow.

The net outward flow of mechanical energy is negligible for streaming velocities small compared with the velocity of light. The force necessary to hold the dipole in place is calculated. This force vanishes when the dipole axis is parallel to the streaming direction, as does the longitudinal electric field.

One interesting non-radiating case which is also treated (for a non-streaming plasma) is the case when the oscillation frequency of the dipole is much less than both the plasma frequencies and the collision frequencies. The characteristic penetration length of the field into the plasma is then given by the classical "skin-depth" formula.

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I. INTRODUCTION

Electromagnetic fields produced by given time-varying current and charge distributions in free space have been the subjects of many calculations, almost from the beginning of electromagnetic theory. Many problems of current interest are complicated by the fact that the fields may be interacting with a plasma or ionized gas. Several problems have been solved in various approximations in recent years for given distributions of charge and current in the presence of a plasma [see, for example, Cohen (1961,62)]. But many gaps remain in our qualitative understanding of the fields to be expected in particular types of plasma situations.

In particular, effects associated with plasma <u>streaming</u> have been investigated relatively little. The calculations which have been done have, to a considerable extent, been concerned with plasmas which, to lowest order, are assumed to be stationary. It is to be expected that some insight into the effects of plasma streaming can be gained by seeking specific solvable problems concerned with radiation into streaming plasmas.

The problem studied in this dissertation is that of the effect of a cold, streaming plasma on the electric and magnetic fields of an oscillating magnetic point dipole. The plasma is unbounded, and streams across the dipole in an arbitrary direction with respect to the dipole's orientation. We assume that the fields of the dipole are sufficiently weak that they impart only a small perturbation to the streaming motion of the plasma. We also assume that the physical dimensions of the dipole are so small that it does not mechanically obstruct the plasma flow; i.e., we idealize the dipole as a point.

Lee and Papas (1965) have considered a similar problem for the oscillating <u>electric</u> dipole. After obtaining an integral representation for the potential four-vector in the rest frame of the dipole, they use the transformation properties of the plasma's dielectric constant to formulate the appropriate Green's function. For streaming velocities small compared with the velocity of light, they conclude that the far zone electromagnetic field is not entirely transverse. As a result, they show that the Poynting vector associated with the electromagnetic dipole is tilted <u>against</u> the direction of plasma flow.

We find qualitatively similar results for the oscillating magnetic dipole, although our results differ considerably in detail. Moreover, we approach the problem from a different point-of-view: through the formalism of a set of linearized, covariant cold plasma equations. These equations contain an equation of continuity and an equation of motion, with temperature neglected, to describe the dynamical properties of the plasma, while Maxwell's equations govern the behavior of the electric and magnetic fields. We treat the magnetic dipole as a small, "external" current source in the sense proposed (for example) by Cohen (1961). That is, we represent the dipole by a miniature current loop which weakly perturbs the streaming plasma. We assume that we may prescribe the current in the dipole at will. The physical reason for the difference between our results and those of Lee and Papas is that their electric dipole possesses, in effect, an oscillating source charge density, whereas our magnetic dipole is a pure divergenceless current source.

In the framework given above, the statement of the problem is relatively straightforward. Yet, the practical difficulties involved in evaluating the formal expressions obtained are far from trivial.

II. THE MAGNETIC DIPOLE

2.1 Statement of the Problem

A cold, collisionless plasma streams across a circular loop of oscillating current (Figure 1). The orientation of the flow vector $\vec{v}_o$ relative to the plane of the loop is arbitrary. We treat the loop as an externally-fixed current source which is unaffected by the plasma. Later, we shall allow the area of the current loop to become vanishingly small and its current to become infinitely large in such a way that we recover a point magnetic dipole.

The current source generates disturbances in the plasma. However, we assume that the source is sufficiently weak and that the disturbances are sufficiently small that linearized cold plasma equations are applicable. We seek analytical expressions for the electric and magnetic fields of the oscillating current loop in the presence of a streaming plasma. This is a relatively simple problem in the absence of a plasma or, for that matter, in the presence of a stationary plasma. But intuition is an unreliable guide to a picture of the fields when the dipole is immersed in a streaming plasma.

2.2 Solution in Wave-Vector, Frequency Space when $\vec{j}_{source_{\parallel}}(\vec{k},\omega) = 0$

We begin with a set of cold plasma equations for the ith species of plasma particle:

Equation of Continuity

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \vec{v}_i) = 0 \tag{1a}$$

Equation of Motion

$$\frac{\partial \vec{v}_i}{\partial t} + \vec{v}_i \cdot \nabla \vec{v}_i = \frac{e_i}{m_i \gamma_i} \left[\vec{E} + \frac{1}{c}(\vec{v}_i \times \vec{B}) - \frac{\vec{v}_i \vec{v}_i}{c^2} \cdot \vec{E}\right] \tag{1b}$$

Maxwell Equations

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{1c}$$

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \tag{1d}$$

$$\nabla \cdot \vec{E} = 4\pi\rho \tag{1e}$$

$$\nabla \cdot \vec{B} = 0 \quad . \tag{1f}$$

The equation of continuity and Maxwell's equations are covariant. The equation of motion is not covariant, although it is correct up to terms of $O(v_i^4/c^4)$, and can be derived from the covariant equation of motion by neglecting fourth order terms in $\frac{v_i}{c}$ (Appendix A). In the equations above, n_i and $\vec{v}_i$ are the number density and velocity, respectively, of the ith species of charge, while

$$\gamma_i = \frac{1}{\sqrt{1 - \frac{v_i^2}{c^2}}} \quad .$$

We have neglected a pressure term in the equation of motion, assuming that the plasma <u>thermal</u> velocities are approximately zero. The current density $\vec{j}$ in equation (1d) is the sum of two parts: an internal plasma current density

$$\vec{j}_p = \sum_i e_i \, n_i \, \vec{v}_i$$

and **an** "external" source current density $\vec{j}_s$, which we will later specify to be that of an oscillating current loop.

Equations (1) are nonlinear; they have not been generally solved. However, we may linearize the equations and obtain a perturbation-theoretic solution about a uniform equilibrium in which the electric and magnetic fields are zero. Since there are no zeroth order electric or magnetic fields in the plasma, we set $\vec{E} = \vec{E}^{(1)}$ and $\vec{B} = \vec{B}^{(1)}$, where the superscript (1) signifies a first-order perturbation. Similarly, if n_{o_i} is the equilibrium number density of the ith plasma component and if $\vec{v}_o$ is the unperturbed streaming velocity common to all components, we may set

$$n_i = n_{o_i} + n_i^{(1)}$$

$$\vec{v}_i = \vec{v}_o + \vec{v}_i^{(1)} ,$$

where n_{o_i} is measured in the rest frame of the streaming plasma. With these substitutions, the cold plasma equations, linearized in the first order perturbations, become

$$\frac{\partial n_i^{(1)}}{\partial t} + n_{o_i} \nabla \cdot \vec{v}_i^{(1)} + \vec{v}_o \cdot \nabla n_i^{(1)} = 0 \qquad (2a)$$

$$\frac{\partial \vec{v}_i^{(1)}}{\partial t} + \vec{v}_o \cdot \nabla \vec{v}_i^{(1)} = \frac{e_i}{m_i \gamma_o} \left[\vec{E}^{(1)} + \frac{1}{c} (\vec{v}_o \times \vec{B}^{(1)}) - \frac{\vec{v}_o \vec{v}_o}{c^2} \cdot \vec{E}^{(1)} \right] \tag{2b}$$

$$\nabla \times \vec{E}^{(1)} = - \frac{1}{c} \frac{\partial \vec{B}^{(1)}}{\partial t} \tag{2c}$$

$$\nabla \times \vec{B}^{(1)} = \frac{4\pi}{c} \vec{j}^{(1)} + \frac{1}{c} \frac{\partial \vec{E}^{(1)}}{\partial t} \tag{2d}$$

where the current density is now

$$\vec{j}^{(1)} = \sum_i e_i \left[n_o \vec{v}_i^{(1)} + n_i^{(1)} \vec{v}_o \right] + \vec{j}_s , \tag{3}$$

the sum of the source current and linearized plasma current densities. We assume that $\sum_i n_{o_i} e_i = 0$, i.e. that there is no zeroth-order charge density in the plasma. Also, since $\vec{v}_o$ is the velocity of all plasma components, there is no zeroth order current density.

Since the differential equations (2) have constant coefficients and are linear in the unknowns $n_i^{(1)}$, $\vec{v}_i^{(1)}$, $\vec{E}^{(1)}$, and $\vec{B}^{(1)}$, we may solve them by Fourier transforming in $\vec{x}$ and t. The Fourier transform of a typical function $f(\vec{x},t)$ is

$$f(\vec{k},\omega) = \frac{1}{(2\pi)^4} \int d\vec{x} \int dt \;\; f(\vec{x},t) e^{-i(\vec{k}\cdot\vec{x} - \omega t)}$$

and a system of differential equations goes over into a system of algebraic equations for the transforms, with $\frac{\partial}{\partial t} \rightarrow -i\omega$ and $\nabla \rightarrow i\vec{k}$ ($\vec{k}$ is the wave vector and ω is the wave frequency for the $\vec{k},\omega$ th Fourier component).

The Fourier transformed Maxwell equations may be combined (as in Appendeix B) to yield the wave equation

$$(\omega^2 - c^2k^2)\, \vec{E}^{(1)} + c^2\, \vec{k}\vec{k} \cdot \vec{E}^{(1)} = -4\pi\, i\omega \sum_i e_i \left[n_o \vec{v}_i^{(1)} + n_i^{(1)} \vec{v}_o \right]$$

$$- 4\pi\, i\omega\, \vec{j}_s\ , \tag{4}$$

where we have used equation (3) for the current density. We may couple the plasma dynamics into the wave equation by solving the Fourier transformed equation of continuity and equation of motion for $n^{(1)}$ and $\vec{v}^{(1)}$ in terms of $\vec{E}^{(1)}$. The result is

$$\vec{v}_i^{(1)} = - \frac{e_i}{m_i \gamma_o (i\omega)} \left\{ \vec{\vec{1}} + \frac{\vec{k}\,\vec{v}_o}{\omega - \vec{k}\cdot\vec{v}_o} - \frac{\omega\, \vec{v}_o \vec{v}_o}{c^2(\omega - \vec{k}\cdot\vec{v}_o)} \right\} \cdot \vec{E}^{(1)} \tag{5a}$$

$$n_i^{(1)} = - \frac{n_{o_i} e_i}{m_i \gamma_o (i\omega)} \ \frac{\vec{k}}{(\omega - \vec{k}\cdot\vec{v}_o)} \cdot \left\{ \vec{\vec{1}} + \frac{\vec{k}\,\vec{v}_o}{(\omega - \vec{k}\cdot\vec{v}_o)} - \frac{\omega\, \vec{v}_o \vec{v}_o}{c^2(\omega - \vec{k}\cdot\vec{v}_o)} \right\} \cdot \vec{E}^{(1)}$$

$$\tag{5b}$$

We derive equations (5a) and (5b) in Appendix A. However, it is worth emphasizing here that we have retained the $\frac{\vec{v}_o}{c} \times \vec{B}^{(1)}$ term in the equation of motion and have eliminated $\vec{B}^{(1)}$ by using equation (2c). The result of substituting equations (5) into the wave equation (4) is

$$(\omega^2 - c^2 k^2)\,\vec{E}^{(1)} + c^2\,\vec{k}\vec{k}\cdot\vec{E}^{(1)}$$

$$- \frac{\sum_i \omega_{p_i}^{\;2}}{\gamma_o}\left[\vec{\vec{I}} + \frac{(\vec{k}\,\vec{v}_o + \vec{v}_o\,\vec{k})}{(\omega - \vec{k}\cdot\vec{v}_o)}\right.$$

$$\left. + \frac{(k^2 - \omega^2/c^2)\,(\vec{v}_o\,\vec{v}_o)}{(\omega - \vec{k}\cdot\vec{v}_o)^2}\right]\cdot\ \vec{E}^{(1)}$$

$$= -\,4\pi\,i\,\omega\,\vec{j}_s \qquad (6)$$

We wish to solve equation (6) for $\vec{E}^{(1)}(\vec{k},\omega)$; we outline below a procedure for accomplishing the solution.

1) Resolve equation (6) into components which are transverse ($\perp$) and parallel ($\parallel$) to the wave vector $\vec{k}$. One of these components will contain only $\vec{j}_{s_\perp}$. The other will contain only $\vec{j}_{s_\parallel}$, which will turn out to be zero for the magnetic dipole.

2) Solve the transverse component for $\vec{v}_o \cdot \vec{E}_\perp^{(1)}$ as a function of $\vec{E}_\parallel^{(1)}$.

3) Solve the parallel component for $\vec{E}_\parallel^{(1)}$ in terms of $\vec{v}_o \cdot \vec{E}_\perp^{(1)}$.

4) Solve simultaneously for $\vec{v}_o \cdot \vec{E}_\perp^{(1)}$ and $\vec{E}_\parallel^{(1)}$ and then solve the transverse component for $\vec{E}_\perp^{(1)}$.

The results for $\vec{j}_{s_\parallel} = 0$ are

$$\vec{E}_\perp^{(1)}(\vec{k},\omega) = -\frac{4\pi\, i\, \omega}{\left[\omega^2 - c^2k^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o}\right]} \left\{ \vec{j}_{s_\perp} - \frac{\sum_i \omega_{p_i}^2 \left(\vec{j}_{s_\perp} \cdot \vec{v}_o\right) \vec{v}_{o_\perp}}{\gamma_o c^2 \left[(\omega - \vec{k}\cdot\vec{v}_o)^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o^3}\right]} \right\} \tag{7}$$

$$\vec{E}_\parallel^{(1)}(\vec{k},\omega) = \frac{-4\pi\, i \left(\sum_i \omega_{p_i}^2\right) \left[1 - \frac{\omega}{c^2}\left(\frac{\vec{k}\cdot\vec{v}_o}{k^2}\right)\right] \vec{k} \left[\vec{j}_{s_\perp} \cdot \vec{v}_o\right]}{\gamma_o \left[\omega^2 - c^2k^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o}\right]\left[(\omega - \vec{k}\cdot\vec{v}_o)^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o^3}\right]} \tag{8}$$

We derive these equations in Appendix C, by implementing the procedure outlined above.

2.3. The Current Transform $\vec{J}_{\text{source}_\perp}(\vec{k},\omega)$ for the Magnetic Dipole

At this point in the development, we require an explicit expression for the Fourier transform of the current loop (Figure 1). In Appendix D we show that for a time dependence $\cos \omega_o t$

$$\vec{J}\,(\vec{k},\omega) = \frac{ic}{16\pi^3}\,[\vec{k} \times \vec{\mu}_o]\,[\,\delta(\omega - \omega_o) \;+\; \delta(\omega + \omega_o)] \quad , \tag{9}$$

where ω_o is the externally-maintained driving frequency. By definition, $\vec{\mu}_o$ is a vector normal to the plane of the current loop with magnitude

$$\left|\vec{\mu}_o\right| = \lim_{\substack{J_o \to \infty \\ a_o \to 0 \\ J_o a_o^2 = \text{constant}}} \frac{\pi\, J_o a_o^2}{c} \quad ,$$

where J_o is the magnitude of the source current and a_o is the radius of the loop. The current transform (9) for the magnetic dipole is purely transverse to the wave vector $\vec{k}$.

2.4 The Electric Field When $\frac{|\vec{v}_o|}{c} \ll 1$

Equations (7) and (8) give the transverse and longitudinal components of the electric field in $(\vec{k},\omega)$ space. The inverse transforms of these equations specify the electric field vector in $(\vec{x},t)$ space. The inverse transform of a function $f(\vec{k},\omega)$ is

$$f(\vec{x},t) = \int d\omega \int d\vec{k}\; f(\vec{k},\omega)\; e^{i(\vec{k}\cdot\vec{x} - \omega t)} .$$

Therefore, using (7) and (8), we have

$$\vec{E}_{\perp}^{(1)}(\vec{x},t) = -(4\pi i) \int d\omega \int d\vec{k} \quad \frac{\omega e^{i(\vec{k}\cdot\vec{x} - \omega t)}}{\left[\omega^2 - c^2k^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o}\right]} \cdot$$

$$\left\{ \vec{j}_{s_\perp} - \frac{\left(\sum_i \omega_{p_i}^2\right)\left(\vec{j}_{s_\perp}\cdot\vec{v}_o\right)\;\vec{v}_{o_\perp}}{\gamma_o\, c^2 \left[\left(\omega-\vec{k}\cdot\vec{v}_o\right)^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o^{\,3}}\right]} \right\} \tag{10}$$

$$\vec{E}_{\parallel}^{(1)}(\vec{x},t) = -(4\pi i)\,\frac{\left(\sum_i \omega_{p_i}^2\right)}{\gamma_o}$$

$$\int d\omega \int d\vec{k}\; \frac{e^{i(\vec{k}\cdot\vec{x}-\omega t)}\left[1-\frac{\omega}{c^2}\left(\frac{\vec{k}\cdot\vec{v}_o}{k^2}\right)\right]\left[\vec{j}_{s_\perp}\cdot\vec{v}_o\right]\vec{k}}{\left[\omega^2 - c^2k^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o}\right]\left[(\omega-\vec{k}\cdot\vec{v}_o)^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o^3}\right]} . \tag{11}$$

In principle, these relations solve the problem, for we may integrate over $\vec{k}$ and ω to recover the electric field in $\vec{x}$ and t. In practice, the integration is difficult due to the factor

$$\left[(\omega - \vec{k}\cdot\vec{v}_o)^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o^3}\right]$$

in the denominator. Separating the denominator into partial fractions circumvents this difficulty for small streaming velocities $\left(\frac{v_o}{c} \ll 1\right)$. Let

$$\frac{1}{\left[\omega^2 - c^2k^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o}\right]\left[(\omega^2 - \vec{k}\cdot\vec{v}_o)^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o^3}\right]}$$

$$= \frac{Mk + N}{\left[\omega^2 - c^2k^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o}\right]} + \frac{Pk + Q}{\left[(\omega - \vec{k}\cdot\vec{v}_o)^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o^3}\right]} \qquad (12)$$

We show in Appendix E that

$$Mk = \frac{2\omega\,(\vec{k}\cdot\vec{v}_o)}{\left[\omega^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o^3}\right]^2 D}$$

$$N = \frac{\left[\omega^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o^3}\right] + \left(\frac{v_o}{c}\right)^2 \cos^2\lambda \left[\omega^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o}\right]}{\left[\omega^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o^3}\right]^2 D}$$

$$Pk = M\left(\frac{v_o}{c}\right)^2 k \cos^2 \lambda$$

$$Q = \frac{1}{\left(\omega^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o}\right)} - \frac{N\left(\omega^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o^3}\right)}{\left(\omega^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o}\right)}$$

where

$$D = \left[1 + 2\left(\frac{v_o}{c}\right)^2 \cos^2 \lambda \frac{\left(\omega^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o}\right)}{\left(\omega^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o^3}\right)} \right.$$

$$- 4\left(\frac{v_o}{c}\right)^2 \cos^2 \lambda \frac{\omega^2\left(\omega^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o}\right)}{\left(\omega^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o^3}\right)^2}$$

$$\left. + \left(\frac{v_o}{c}\right)^4 \cos^4 \lambda \frac{\left(\omega^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o}\right)^2}{\left(\omega^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o^3}\right)^2} \right]$$

and $\cos \lambda = \frac{\vec{k} \cdot \vec{v}_o}{\left|\vec{k}\right| \left|\vec{v}_o\right|}$.

We may expand the denominator D for small $\frac{v_o}{c}$ $(\gamma_o \simeq 1)$, provided we restrict ourselves to velocities $\vec{v}_o$ and frequencies ω such that the term

$$-4\left(\frac{v_o}{c}\right)^2 \cos^2\lambda \frac{\omega^2}{\left(\omega^2 - \sum_i \omega_{p_i}^2\right)}$$

is sufficiently small. That is, we exclude those frequencies within some small range ($\pm\,\epsilon$, say) of $\left(\sum_i \omega_{p_i}^2\right)^{1/2}$. If we perform the suggested expansion and disregard all terms of order $\left(\frac{v_o}{c}\right)^2$ and above, the coefficients P and Q in (12) vanish and we have

$$\frac{1}{\left[\omega^2 - c^2k^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o}\right]\left[(\omega - \vec{k}\cdot\vec{v}_o)^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o^3}\right]}$$

$$\simeq \frac{2\omega\,(\vec{k}\cdot\vec{v}_o) + \left(\omega^2 - \sum_i \omega_{p_i}^2\right)}{\left[\left(\omega^2 - \sum_i \omega_{p_i}^2\right)\right]^2\left[\omega^2 - c^2k^2 - \sum_i \omega_{p_i}^2\right]}$$

But equation (11) and the "distortion" term in equation (10) are already at least of order $\frac{v_o}{c}$. Therefore, if we again disregard terms of order $\left(\frac{v_o}{c}\right)^2$, we find that

$$\vec{E}_\perp^{(1)}(\vec{x},t) \simeq -(4\pi i) \int d\omega \int d\vec{k} \; \frac{\omega \, e^{i(\vec{k}\cdot\vec{x} - \omega t)} \vec{J}_{s_\perp}}{\left[\omega^2 - c^2k^2 - \sum_i \omega_{p_i}^2\right]} \tag{13}$$

$$\vec{E}_{\parallel}^{(1)}(\vec{x},t) \simeq -(4\pi i) \left(\sum_i \omega_{p_i}^2\right) \int d\omega \int d\vec{k} \; \frac{\vec{k}\, e^{i(\vec{k}\cdot\vec{x} - \omega t)} (\vec{J}_{s_\perp} \cdot \vec{v}_o)}{\left(\omega^2 - \sum_i \omega_{p_i}^2\right)\left[\omega^2 - c^2k^2 - \sum_i \omega_{p_i}^2\right]} \tag{14}$$

which are correct up to terms of order $\left(\frac{v_o}{c}\right)^2$. The transverse field (Eq. 13) is the same as the zero-streaming limit, but the longitudinal field (Eq. 14) is new. Using (9) for the current transform and removing the $\vec{k}$ operators from the integrand ($k \rightarrow -i\nabla$), we may write (13) and (14) as

$$\vec{E}_{\perp}^{(1)}(\vec{x},t) = \frac{i}{4\pi^2 c} \quad \nabla \quad \times$$

$$\left\{ \vec{\mu}_o \int d\omega \int d\vec{k} \quad \frac{\omega\,[\delta(\omega-\omega_o) + \delta(\omega+\omega_o)]e^{i(\vec{k}\cdot\vec{x}-\omega t)}}{\left[k^2 - \frac{1}{c^2}\left(\omega^2 - \sum_i \omega_{p_i}^2\right)\right]} \right\} \tag{15}$$

$$\vec{E}_{\parallel}^{(1)}(\vec{x},t) = \frac{\left(\sum_i \omega_{p_i}^2\right)}{4\pi^2}$$

$$\nabla \left\{ \frac{\vec{v}_o}{c} \cdot \left[\nabla \times \left(\vec{\mu}_o \int d\omega \int d\vec{k} \, \frac{[\delta(\omega-\omega_o) + \delta(\omega+\omega_o)]e^{i(\vec{k}\cdot\vec{x}-\omega t)}}{\left[\omega^2 - \sum_i \omega_{p_i}^2\right]\left[k^2 - \frac{1}{c^2}\left(\omega^2 - \sum_i \omega_{p_i}^2\right)\right]}\right)\right]\right\} \tag{16}$$

The transverse electric field is undistorted for small streaming velocities; one would obtain the same expression for an oscillating magnetic dipole in a stationary plasma. On the other hand, the longitudinal electric field is directly proportional to the plasma's streaming velocity $\vec{v}_o$; $\vec{E}_\parallel^{(1)}$ vanishes when $\vec{v}_o = 0$. This is despite the fact that the source current is purely transverse.

We now perform the $\vec{k}$ and ω integrations (Appendix F). The results, for

$$\left|\omega_o\right| \geq \left(\sum_i \omega_{p_i}^2\right)^{1/2} + \left|\epsilon\right| ,$$

are

$$\vec{E}_\perp^{(1)}(\vec{x},t) = -\frac{\omega_o}{c}\nabla \times \left\{ \frac{\vec{\mu}_o}{\left|\vec{x}\right|} \sin\left(a\left|\vec{x}\right| - \omega_o t\right)\right\} \qquad (17)$$

$$\vec{E}_\parallel^{(1)}(\vec{x},t) = \frac{\sum_i \omega_{p_i}^2}{\left(\omega_o^2 - \sum_i \omega_{p_i}^2\right)}\left\{\nabla\left[\frac{\vec{v}_o}{c}\cdot\left(\nabla \times \vec{\mu}_o \frac{\cos\left(a\left|\vec{x}\right| - \omega_o t\right)}{\left|\vec{x}\right|}\right)\right]\right\},$$

(18)

where $$a = \frac{\omega_o}{c}\left[1 - \frac{\sum_i \omega_{p_i}^2}{\omega_o^2}\right]^{\frac{1}{2}} .$$

For $\left(\sum_i \omega_{p_i}^2\right)^{\frac{1}{2}} - \left|\epsilon\right| > \left|\omega_o\right| > 0$, the fields are

$$\vec{E}_{\perp}^{(1)}(\vec{x},t) = \left(\frac{\omega_o}{c}\right)\nabla \times \left\{ \frac{\vec{\mu}_o\, e^{-b\left|\vec{x}\right|}}{\left|\vec{x}\right|} \sin \omega_o t \right\} \tag{19}$$

$$\vec{E}_{\parallel}^{(1)}(\vec{x},t) = -\frac{\sum_i \omega_{p_i}^2 \cos \omega_o t}{\left(\sum_i \omega_{p_i}^2 - \omega_o^2\right)} \nabla \left\{ \frac{\vec{v}_o}{c} \cdot \nabla \times \frac{\vec{\mu}_o\, e^{-b\left|\vec{x}\right|}}{\left|\vec{x}\right|} \right\} , \tag{20}$$

where $b = \dfrac{\left(\sum_i \omega_{p_i}^2\right)^{\frac{1}{2}}}{c} \left[1 - \dfrac{\omega_o^2}{\sum_i \omega_{p_i}^2}\right]^{\frac{1}{2}}$.

Our results thus far are independent of coordinates. We now choose the coordinate configuration in Figure 2. Let the y axis of an (xyz) Cartesian system point in the direction of the streaming velocity $\vec{v}_o$; the xz plane is normal to the direction of flow. We place the dipole at the origin of the coordinate system and let the plasma stream past. The polar angle θ_m, measured from the z axis, and the azimuthal angle φ_m, measured from the x axis, define the angular orientation of the dipole. The spherical coordinates (r,θ,φ) specify the position of an observer relative to the origin. In terms of the parameters and coordinates defined above, the components of the transverse electric field, for $|\omega_o| > \left(\sum_i \omega_{p_i}^2\right)^{1/2}$, are

$$E_{\perp r} = 0$$

$$E_{\perp\theta} = -\left(\frac{\mu_o\omega_o}{c}\right) \sin\theta_m \sin(\varphi-\varphi_m) \left[\frac{a\cos(ar-\omega_o t)}{r} - \frac{\sin(ar-\omega_o t)}{r^2}\right]$$

$$E_{\perp\varphi} = \left(\frac{\mu_o\omega_o}{c}\right)\left\{[\sin\theta\cos\theta_m - \cos\theta\sin\theta_m\cos(\varphi-\varphi_m)] \cdot \left[\frac{a\cos(ar-\omega_o t)}{r} - \frac{\sin(ar-\omega_o t)}{r^2}\right]\right\}. \tag{21}$$

For $\left(\sum_i \omega_{p_i}^2\right)^{\frac{1}{2}} > \left| \omega_o \right| > 0$,

$$E_{\perp_r} = 0$$

$$E_{\perp_\theta} = -\left(\frac{\mu_o \omega_o}{c}\right) e^{-br} \sin \theta_m \sin (\varphi - \varphi_m) \left[\frac{b}{r} + \frac{1}{r^2}\right] \sin \omega_o t$$

$$E_{\perp_\varphi} = \left(\frac{\mu_o \omega_o}{c}\right) e^{-br} [\sin\theta \cos\theta_m - \cos\theta \sin\theta_m \cos(\varphi-\varphi_m)] \cdot \left[\frac{b}{r} + \frac{1}{r^2}\right] \sin \omega_o t \ . \tag{22}$$

The components of the longitudinal electric field, for $\left| \omega_o \right| > \left(\sum_i \omega_{p_i}^2\right)^{1/2}$, are

$$E_{H\,r} = \frac{\mu_o \left(\sum_i \omega_{p_i}^{\;2}\right)}{\left(\omega_o^{\;2} - \sum_i \omega_{p_i}^{\;2}\right)} \left(\frac{v_o}{c}\right) [\sin\theta \cos\varphi \cos\theta_m - \cos\theta \sin\theta_m \cos\varphi_m] \cdot$$

$$\left[\frac{a^2 \cos(ar-\omega_o t)}{r} - \frac{2a \sin(ar-\omega_o t)}{r^2} - \frac{2 \cos(ar-\omega_o t)}{r^3}\right]$$

$$E_{H\,\theta} = \frac{\mu_o \left(\sum_i \omega_{p_i}^{\;2}\right)}{\left(\omega_o^{\;2} - \sum_i \omega_{p_i}^{\;2}\right)} \left(\frac{v_o}{c}\right) [\cos\theta \cos\varphi \cos\theta_m + \sin\theta \sin\theta_m \cos\varphi_m] \cdot$$

$$\left[\frac{a \sin(ar-\omega_o t)}{r^2} + \frac{\cos(ar-\omega_o t)}{r^3}\right]$$

$$E_{H\,\varphi} = - \frac{\mu_o \left(\sum_i \omega_{p_i}^{\;2}\right)}{\left(\omega_o^{\;2} - \sum_i \omega_{p_i}^{\;2}\right)} \left(\frac{v_o}{c}\right) \sin\varphi \cos\theta_m$$

$$\left[\frac{a \sin(ar-\omega_o t)}{r^2} + \frac{\cos(ar-\omega_o t)}{r^3}\right] . \qquad (23)$$

For $\left(\sum_i \omega_{p_i}^2\right)^{\frac{1}{2}} > \left| \omega_o \right| > 0$, the components of $\vec{E}_H$ are

$$E_{Hr} = \frac{\mu_o \left(\sum_i \omega_{p_i}^2\right)}{\left(\sum_i \omega_{p_i}^2 - \omega_o^2\right)} \left(\frac{v_o}{c}\right) [\sin\theta \cos\varphi \cos\theta_m - \cos\theta \sin\theta_m \cos\varphi_m] \cdot$$

$$e^{-br} \left[\frac{b^2}{r} + \frac{2b}{r^2} + \frac{2}{r^3}\right] \cos \omega_o t$$

$$E_{H\theta} = - \frac{\mu_o \left(\sum_i \omega_{p_i}^2\right)}{\left(\sum_i \omega_{p_i}^2 - \omega_o^2\right)} \left(\frac{v_o}{c}\right) [\cos\theta \cos\varphi \cos\theta_m + \sin\theta \sin\theta_m \cos\varphi_m] \cdot$$

$$e^{-br} \left[\frac{b}{r^2} + \frac{1}{r^3}\right] \cos \omega_o t$$

$$E_{H\varphi} = \frac{\mu_o \left(\sum_i \omega_{p_i}^2\right)}{\left(\sum_i \omega_{p_i}^2 - \omega_o^2\right)} \left(\frac{v_o}{c}\right) (\sin\varphi \cos\theta_m) e^{-br} \left[\frac{b}{r^2} + \frac{1}{r^3}\right] \cos \omega_o t.$$

(24)

2.5 The Magnetic Field $\vec{B}(\vec{x},t)$ when $\frac{|\vec{v}_o|}{c} << 1$

We use the Maxwell equation

$$\nabla \times \vec{E}^{(1)} = -\frac{1}{c}\frac{\partial \vec{B}^{(1)}}{\partial t}$$

to find the magnetic field $\vec{B}^{(1)}$ in the plasma; there is, of course, no magnetic field associated with $\vec{E}_{\parallel}^{(1)}$, which is derivable from an electrostatic potential. The results, for

$$|\omega_o| > \left(\sum_i \omega_{p_i}^2\right)^{\frac{1}{2}} \quad , \text{ are}$$

$$B_r = 2\mu_o \left\{ [\cos\theta \cos\theta_m + \sin\theta \sin\theta_m \cos(\varphi - \varphi_m)] \cdot \left[\frac{a \sin(ar-\omega_o t)}{r^2} + \frac{\cos(ar-\omega_o t)}{r^3}\right]\right\}$$

$$B_\theta = -\mu_o \left\{ [\sin\theta \cos\theta_m - \cos\theta \sin\theta_m \cos(\varphi-\varphi_m)] \cdot \left[\frac{a^2 \cos(ar-\omega_o t)}{r} - \frac{a \sin(ar-\omega_o t)}{r^2} - \frac{\cos(ar-\omega_o t)}{r^3}\right]\right\}$$

$$B_\varphi = -\mu_o \sin\theta_m \sin(\varphi-\varphi_m) \left[\frac{a^2\cos(ar-\omega_o t)}{r} - \frac{a \sin(ar-\omega_o t)}{r^2} - \frac{\cos(ar-\omega_o t)}{r^3}\right] . \tag{25}$$

For $\left(\sum_i \omega_{p_i}^2\right)^{\frac{1}{2}} > \left| \omega_o \right| > 0$,

$$B_r = 2\mu_o e^{-br} \left[\frac{b}{r^2} + \frac{1}{r^3} \right] [\cos\theta \cos\theta_m + \sin\theta \sin\theta_m \cos(\varphi - \varphi_m)] \cos \omega_o t$$

$$B_\theta = \mu_o e^{-br} \left[\frac{b^2}{r} + \frac{b}{r^2} + \frac{1}{r^3} \right] [\sin\theta\cos\theta_m - \cos\theta\sin\theta_m \cos(\varphi - \varphi_m)] \cos \omega_o t$$

$$B_\varphi = \mu_o e^{-br} \left[\frac{b^2}{r} + \frac{b}{r^2} + \frac{1}{r^3} \right] [\sin \theta_m \sin (\varphi - \varphi_m)] \cos \omega_o t \text{ .} \qquad (26)$$

When $\left(\sum_i \omega_{p_i}^2\right)^{\frac{1}{2}} > \left| \omega_o \right| > 0$, the electric and magnetic fields are exponentially damped in space and the transverse electric and magnetic fields are 90° out of phase with each other in time.

III. INTERPRETATION OF THE SOLUTION

3.1 Field Distortion: The Longitudinal Electric Field

Equations (21) through (26) specify the electric and magnetic fields which an oscillating magnetic dipole induces in a streaming plasma. The equations hold for arbitrary orientations of $\vec{\mu}_o$ with respect to $\vec{v}_o$, subject to the restriction that $\frac{v_o}{c} << 1$.

Some observations are in order. First (and this is perhaps the most interesting result of our study), it is apparent that an oscillating magnetic dipole, in the presence of a streaming plasma, excites a longitudinal electric field which vanishes when $\vec{v}_o = 0$. The longitudinal electric field is coupled with the transverse electromagnetic field and both fields oscillate at the driving frequency of the current source. Yet, the source producing the longitudinal electric field is purely transverse. Second, it is clear that for $\frac{v_o}{c} << 1$ the transverse electric and magnetic fields are essentially undistorted by the streaming plasma. Mathematically, this results from the fact that $\vec{J}_s(\vec{k}, \omega)$ for the magnetic dipole is purely transverse. For the electric dipole, on the other hand, $\vec{j}_s(\vec{k},\omega)$ has both a perpendicular <u>and</u> a parallel component. The development in Appendix C predicts that when $\vec{j}_{s_\parallel}$ is non-zero, the transverse electric field will sustain an order $\frac{v_o}{c}$ distortion. The results of Lee and Papas (1965) confirm this prediction. Finally, we note that for $\vec{\mu}_o \parallel \vec{v}_o$ ($\theta_m = \varphi_m = 90°$ in equations 21 through 26), there is no longitudinal electric field; an $\vec{E}_\parallel$ only exists when $\vec{\mu}_o$ has a non-zero component perpendicular to the streaming velocity $\vec{v}_o$.

3.2 Power Flow: The Skewed Poynting Vector

At great distances from the dipole, only the order $\frac{1}{r}$ terms in the electric and magnetic field equations are significant. We find that the existence -- or perhaps we should say the survival -- of a longitudinal electric field component at large r skews the Poynting vector,

$$\vec{S} = \frac{c}{4\pi}(\vec{E} \times \vec{B}) ,$$

away from the radial orientation that it would have in a purely transverse electromagnetic field. Using equations (21), (23), and (25) for $\vec{\mu}_o \perp \vec{v}_o$ and $\left|\omega_o\right| > \left(\sum_i \omega_{p_i}^2\right)^{1/2}$, we calculate

$$S_r = \frac{\mu_o^2}{4\pi c^3 r^2} \left(\omega_o^4 n_a^3\right) \sin^2\theta \cos^2(ar-\omega_o t)$$

$$S_\theta = 0$$

$$S_\varphi = -\frac{\mu_o^2}{4\pi c^3 r^2} \left(\frac{v_o}{c}\right) (\omega_o n_a)^2 \left(\sum_i \omega_{p_i}^2\right) \sin^2\theta \cos\varphi \cos^2(ar-\omega_o t), \quad (27)$$

where $n_a = \left(1 - \frac{\sum_i \omega_{p_i}^2}{\omega_o^2}\right)^{\frac{1}{2}}$.

The radial component of $\vec{S}$ is the same as it is in a stationary plasma; the azimuthal component of $\vec{S}$ is dependent on $\vec{v}_o$ and points in the "upstream" direction (Figure 3). Lee and Papas find a similar result for the electric dipole: The Poynting vector is tilted _against_ the direction of flow.

3.3 Power Balance: Mechanical Energy Transmission

Field [1956] has derived a generalization of Poynting's theorem for a _hot_, _non-streaming_ plasma. He shows that for non-zero thermal velocities there may be a flux of mechanical energy -- in addition to the flux of electromagnetic energy -- across a surface enclosing the source. We derive below an expression for the total energy flux in a _cold_, _streaming_ plasma.

In order to do this, we return to the cold plasma equations of Section 2.2. Using the two Maxwell curl equations, we may easily show that

$$\nabla \cdot \vec{S} + \frac{\partial \upsilon}{\partial t} = - \vec{E} \cdot \vec{j}_p - \vec{E} \cdot \vec{j}_s ,$$

where $\vec{S} = \frac{c}{4\pi} (\vec{E} \times \vec{B})$ is the Poynting vector; $\upsilon = \frac{1}{8\pi} (E^2 + B^2)$ is the electromagnetic energy density; $\vec{j}_s$ is the external source current density; and $\vec{j}_p = \sum_i e_i \, n_i \, \vec{v}_i$ is the internal plasma current density. Next, we use the equation of motion (with $\gamma_i = 1$) and the equation of continuity to express $\left(\sum_i e_i \, n_i \, \vec{v}_i\right) \cdot \vec{E}$ as the divergence of a mechanical power flow plus the time derivative of a mechanical energy density. The dot product of the momentum density, $m_i n_i \vec{v}_i$, and the force equation for the ith species is

$$n_i m_i \vec{v}_i \cdot \frac{\partial}{\partial t} \vec{v}_i + m_i n_i \vec{v}_i \cdot (\vec{v}_i \cdot \nabla) \vec{v}_i = e_i n_i \vec{v}_i \cdot \vec{E} , \qquad (28)$$

since $\vec{v}_i \cdot \frac{\vec{v}_i}{c} \times \vec{B} = 0.$

Consider the second term in equation (28). Using appropriate vector identities, we find that

$$n_i \vec{v}_i \cdot (\vec{v}_i \cdot \nabla)\vec{v}_i = \frac{1}{2} n_i \vec{v}_i \cdot \nabla (v_i^2)$$

$$= \frac{1}{2} \left\{ \nabla \cdot [(n_i v_i^2) \vec{v}_i] - v_i^2 [\nabla \cdot (n_i \vec{v}_i)] \right\} .$$

But by the equation of continuity

$$\nabla \cdot (n_i \vec{v}_i) = - \frac{\partial n_i}{\partial t}$$

Therefore,

$$m_i n_i \vec{v}_i \cdot (\vec{v}_i \cdot \nabla)\vec{v}_i = \frac{1}{2} m_i \left\{ \nabla \cdot [(n_i v_i^2)\vec{v}_i] + v_i^2 \frac{\partial n_i}{\partial t} \right\} \tag{29}$$

Now, consider the first term in equation (28):

$$m_i n_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial t} = \frac{1}{2} m_i \left\{ \frac{\partial}{\partial t} [n_i v_i^2] - v_i^2 \frac{\partial n_i}{\partial t} \right\} . \tag{30}$$

The $v_i^2 \frac{\partial n_i}{\partial t}$ terms in (29) and (30) cancel. Consequently,

$$\vec{J}_p \cdot \vec{E} = \sum_i \left\{ \frac{\partial}{\partial t} [\frac{1}{2} n_i m_i v_i^2] + \nabla \cdot [(\frac{1}{2} n_i m_i v_i^2) \vec{v}_i] \right\}$$

and the Poynting's theorem appropriate to a cold, streaming plasma is

$$\nabla \cdot [\vec{S} + \sum_i (\frac{1}{2} n_i m_i v_i^2)\vec{v}_i] + \frac{\partial}{\partial t} [\upsilon + \sum_i (\frac{1}{2} n_i m_i v_i^2)] = -\vec{E} \cdot \vec{J}_s \qquad (31)$$

We will neglect the time derivative of the energy density in this equation; for sinusoidal oscillations, its time average is zero.

Equation (31) suggests the possibility that mechanical energy transmission may be associated with the longitudinal electric field. Let us examine this idea. First, we enclose the dipole within a large sphere of radius r and form the volume integral of equation (31) -- without the energy density term.

Using the divergence theorem for the integral on the left, we find that

$$\oiint da\, \hat{n} \cdot [\vec{S} + \sum_i (\tfrac{1}{2} n_i m_i v_i^{\,2})\, \vec{v}_i] = - \int_V d^3x\, (\vec{E} \cdot \vec{j}_s) \tag{32}$$

where the volume integral on the right is the external power input. The surface integral is equivalent to

$$\oiint da\, [S_r + \sum_i (\tfrac{1}{2} n_i m_i v_i^{\,2})\, v_{i_r}] \tag{33}$$

since the unit vector $\hat{n}$ is parallel to $\hat{r}$.

In our linearized version of the problem, $\vec{v}_i = \vec{v}_o + \vec{v}_i^{\,(1)}$ and $n_i = n_{o_i} + n_i^{(1)}$. We may calculate both $\vec{v}_i^{\,(1)}(\vec{x},t)$ and $n_i^{(1)}(\vec{x},t)$ with the results of Section II. For $\vec{\mu}_o \perp \vec{v}_o$ and $\left|\omega_o\right| > \left(\sum_i \omega_{p_i}^{\,2}\right)^{\frac{1}{2}}$,

$$\vec{v}_i^{\,(1)}(\vec{x},t) \simeq -\left(\frac{e_i}{m_i}\right)\left(\frac{\mu_o\omega_o}{c^2\, r}\right) \sin\,(ar-\omega_o t)\,.$$

$$\sin\theta\left\{\left[\left(\frac{v_o}{c}\right)\cos\varphi\right]\hat{r} + [n_a]\,\hat{\varphi}\right\}$$

$$\simeq \left(v_{i\ r}^{(1)}\right)\hat{r} + \left(v_{i\ \varphi}^{(1)}\right)\hat{\varphi} \tag{34}$$

$$n_i^{(1)}(\vec{x},t) \simeq -\left(\frac{n_o e_i}{m_i}\right)\left(\frac{v_o}{c}\right)\left(\frac{\mu_o}{c^2}\right)\left(\frac{\omega_o}{c}\ n_a\right)\ \frac{\sin\theta\cos\varphi}{r}\sin(ar-\omega_o t) \tag{35}$$

where we have retained only those terms of order $(\frac{1}{r})$ in distance.

When we linearize the mechanical power part of (33) and use (34) and (35) for the first order perturbations, we find that the only terms which survive the surface integration, as r becomes infinitely large, are

$$\frac{1}{2}\sum_i\left[2n_{o_i}m_i\left(v_{o_\varphi}\ v_{i_\varphi}^{(1)}\right)\ v_{i_r}^{(1)}\right],$$

$$\frac{1}{2}\sum_i\left[\left(n_i^{(1)}\ m_i\ v_o^2\right)\ \ v_{i_r}^{(1)}\right]$$

and $$\frac{1}{2}\sum_i\left[2n_i^{(1)}m_i\left(v_{o_r}\ v_{i_r}^{(1)}\right)\ \ v_{o_r}\right].$$

(We note, immediately, that these terms vanish when $\left|\vec{v}_o\right| = 0$ -- as expected.) The latter two terms are of order $\frac{v_o^4}{c^4}$ and may be neglected. However, the first term is of order $\frac{v_o^2}{c^2}$ since $v_{i_\varphi}^{(1)}$ is not a function of v_o. The time-averaged mechanical power flow is , as a result,

$$\oint\!\!\oint da \left[\sum_i \left(\frac{1}{2} n_i m_i v_i^2\right) v_{i_r} \right] \Bigg|_{\text{time average}} =$$

$$\frac{1}{6} \left(\sum_i \omega_{p_i}^2\right) \left(\frac{\mu_o^2}{c^2}\right) \left(\frac{\omega_o^2 n_a}{c}\right) \left(\frac{v_o}{c}\right)^2 \quad . \qquad (36)$$

Therefore, in principle, there is an outward flow of mechanical energy when the dipole oscillates in the presence of a streaming plasma, although the energy transfer is negligible in the limit of small streaming velocities. In addition, since the Poynting flux is accurate only through terms of $O(^{v}o/c)$, the expression (36) is of no use in verification of the conservation laws.

We compare the outward flux of mechanical energy with the outward flux of electromagnetic energy, as given by the first term in equation (33). The time-averaged electromagnetic power <u>output</u> is

$$\oiint da \, S_r \Bigg|_{\text{time average}} = \frac{1}{3} \frac{\mu_o^2 \, \omega_o^4 \, n_a^3}{c^3} \tag{37}$$

for small $\left|\frac{v_o}{c}\right|$, regardless of whether the plasma is streaming or stationary. It can be shown that performing the volume integration on the right hand side of (32), with the fields in (21), also gives equation (37) for the power <u>input</u>. The additional flux of mechanical power, as given by equation (36) is $O(v_o^2/c^2)$ and thus negligible.

The case $|\omega_o| < \left(\sum_i \omega_{p_i}^2\right)^{\frac{1}{2}}$ is a case of not much interest in this context. Both the time averaged radial Poynting flux and the time averaged volume integral of $\vec{E}^{(1)} \cdot \vec{j}_s$ can be easily shown from equations (22) and (26) to be identically zero. No electromagnetic energy is radiated into the plasma.

<u>3.4 Momentum Conservation: The Mechanical Force on the Dipole</u>

The streaming plasma exerts a mechanical force on the dipole -- a force which tends to push the dipole "downstream", in the direction of flow. Let $\overleftrightarrow{T}$ be the Maxwell stress tensor:

$$\overleftrightarrow{T} = \frac{1}{4\pi} [\vec{E}\vec{E} + \vec{B}\vec{B} - \frac{1}{2} \overleftrightarrow{I} (E^2 + B^2)] \quad .$$

We may calculate the force which acts on the dipole by enclosing it once again within a sperical boundary surface S and evaluating the integral $\oiint \hat{n} \cdot \overleftrightarrow{T}\, da$ over S, where $\hat{n}da$ is an outwardly directed element of surface area. We have

$$\oiint_S \hat{n} \cdot \overleftrightarrow{T}\, da = \frac{1}{4\pi} \oiint r^2 \sin\theta \, d\theta \, d\varphi \Big\{ E_r (E_r\hat{r} + E_\theta\hat{\theta} + E_\varphi\hat{\varphi})$$

$$+ B_r (B_r\hat{r} + B_\theta\hat{\theta} + B_\varphi\hat{\varphi}) - \frac{1}{2}\hat{r}[(E_r^{\,2} + E_\theta^{\,2} + E_\varphi^{\,2})$$

$$+ (B_r^{\,2} + B_\theta^{\,2} + B_\varphi^{\,2})]\Big\} \qquad (38)$$

This time, in order to perform the areal integration, we must allow for the fact that the unit vectors $\hat{r}$, $\hat{\theta}$, and $\hat{\varphi}$ are, themselves, functions of their angular position on the surface of the sphere. Therefore, we express $\hat{r}$, $\hat{\theta}$, and $\hat{\varphi}$ in terms of the fixed Cartesian unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, where $\hat{\jmath}$ points in the direction of flow.

$$\hat{r} = (\sin\theta \cos\varphi)\,\hat{\imath} + (\sin\theta \sin\varphi)\,\hat{\jmath} + (\cos\theta)\,\hat{k}$$

$$\hat{\theta} = (\cos\theta \cos\varphi)\,\hat{\imath} + (\cos\theta \sin\varphi)\,\hat{\jmath} - (\sin\theta)\,\hat{k}$$

$$\hat{\varphi} = (-\sin\varphi)\,\hat{\imath} + (\cos\varphi)\,\hat{\jmath} \tag{39}$$

Using (39) and the <u>complete</u> field components (21), (23), and (25), we find that for $\vec{\mu}_o \perp \vec{v}_o$ and $\left|\omega_o\right| > \left(\sum_i \omega_{p_i}^2\right)^{\frac{1}{2}}$ the time-averaged value of (38) is

$$\oiint_s \hat{n} \cdot \overleftrightarrow{T}\, da \Bigg|_{\text{time average}} = \left[\frac{\mu_o^2\, \omega_o^2 \left(\sum_i \omega_{p_i}^2\right) n_a}{6c^4}\right] \frac{v_o}{c}\, \hat{\jmath} \tag{40}$$

The expression above is equal to the time average of $\frac{d\vec{P}}{dt}$, where $\vec{P}$ is the mechanical momentum of the particles (and the dipole) plus the electromagnetic momentum of the fields within the volume of the sphere. Specifically,

$$\frac{d\vec{P}_{\text{mechanical}}}{dt} = \int_V \left[\rho\vec{E} + \frac{1}{c}(\vec{J} \times \vec{B})\right] d^3x \ ,$$

$$\vec{P}_{\text{field}} = \frac{1}{4\pi c} \int_V (\vec{E} \times \vec{B})\, d^3x \ .$$

But the time average of the time derivative of $\vec{P}_{\text{field}}$ is zero. Moreover, (40) is independent of r, the radius of the sphere. In the limit as $r \to 0$, the sphere encloses only the dipole $\vec{\mu}_o$ at the origin; it no longer contains any particles of plasma. Hence, we may interpret (40) as the effective mechanical force which the streaming plasma exerts on the dipole. The force is parallel to the direction of flow (Figure 4), and it vanishes when either $\vec{v}_o = 0$ or $\left(\sum_i \omega_{p_i}^2\right)^{\frac{1}{2}}$ is zero. An equal and opposite force must be supplied externally to keep the dipole in place. For $\vec{\mu}_o$ parallel to $\vec{v}_o$, the force on the dipole vanishes. The mechanical work per second necessary to drag the dipole with velocity $\vec{v}_o$ through a quiescent plasma would be given by dotting Equation (40) with $\vec{v}_o$.

3.5 Low Frequency Limit: The Penetration Depth when $\vec{v}_o = 0$

There is one special case in which the collisional frequencies ν_j do play an important role in determining the fields:

$\left(\Sigma_j \omega_{p_j}^2\right)^{\frac{1}{2}} >> \text{all } \nu_j >> \omega_o$ (i.e., the "d.c." limit). Equations (22), (24), and (26) show that the fields of the dipole fall off exponentially with distance when

$$\left(\sum_i \omega_{p_i}^2\right)^{\frac{1}{2}} > \left| \omega_o \right| > 0 .$$

The damping decrement b in these expressions, where

$$b = \frac{\left(\sum_i \omega_{p_i}^2\right)^{\frac{1}{2}}}{c} \left(1 - \frac{\omega_o^2}{\sum_i \omega_{p_i}^2}\right)^{\frac{1}{2}} ,$$

applies to the limiting case of a collisionless plasma with the restriction that $\left| \omega_{p_j} - \epsilon \right| > \omega_o >> \text{all } \nu_j$ <u>before</u> the ν_j are set equal to zero. As shown in Appendix F, the damping decrement with a non-zero collision frequency is

$$b = \left(\sqrt{\rho}\right) = [\alpha^2 + \beta^2]^{\frac{1}{4}} , \qquad \text{where}$$

$$\alpha = \frac{\omega_o^2}{c^2} \left[1 - \sum_j \frac{\omega_{p_j}^2}{(\omega_o^2 + \nu_j^2)}\right]$$

$$\beta = \frac{\omega_o}{c^2} \left[\sum_j \frac{\omega_{p_j}^2 \nu_j}{(\omega_o^2 + \nu_j^2)}\right]$$

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Radiation from an Oscillating Magnetic
Dipole in a Streaming Plasma
by JOHN E. BERGESON

Page 2, last line:

Replace word "dipole" with "field" to read as follows:

"magnetic field is tilted against the direction of plasma flow."

Page 9, the line appearing just above equation (5a) should read as follows:

"for $n_i^{(1)}$ and $\vec{v}_i^{(1)}$ in terms of $\vec{E}^{(1)}$. The result is".

Page 56:

In the third and fourth lines from the bottom of the page Appendix A should be substituted for Appendix B to read as follows:

"This is the wave equation, with the expression derived in Appendix A substituted for the plasma current density $\vec{j}_p$. For conciseness,".

The preceding expression is applicable to a slowly streaming plasma when $\omega_o >> \nu_j$. When ω_o is smaller than the ν_j and the streaming velocity is zero, the damping decrement becomes

$$b = \frac{1}{\sqrt{2}} \left\{ \frac{\omega_o^4}{c^4} \left[\sum_j \frac{\omega_{p_j}^2}{(\omega_o^2 + \nu_j^2)} - 1 \right]^2 + \frac{\omega_o^2}{c^4} \left[\sum_j \frac{\omega_{p_j}^2 \, \nu_j}{(\omega_o^2 + \nu_j^2)} \right]^2 \right\}^{\frac{1}{4}} . \tag{41}$$

This vanishes in the limit as ω_o goes to zero. Consequently, the $\frac{1}{r}$ and $\frac{1}{r^2}$ terms in the field equations disappear and there is no longer any exponential damping. In fact, we recover the $\vec{B}$ field of a static point magnetic dipole immersed in a plasma (which is the same as the vacuum field, notice):

$$B_r = \frac{2\mu_o}{r^3} \left[\cos\theta \cos\theta_m + \sin\theta \sin\theta_m \cos(\varphi - \varphi_m)\right]$$

$$B_\theta = \frac{\mu_o}{r^3} \left[\sin\theta \cos\theta_m - \cos\theta \sin\theta_m \cos(\varphi - \varphi_m)\right]$$

$$B_\varphi = \frac{\mu_o}{r^3} \left[\sin\theta_m \sin(\varphi - \varphi_m)\right] . \tag{42}$$

The transverse electric field vanishes when $\omega_o = 0$ (as it must) and the longitudinal electric field disappears when $\vec{v}_o = 0$.

We may use equation (41) to derive an expression for the distance away from the dipole at which the fields are damped to $\frac{1}{e}$ of their initial amplitude. For very low frequencies, the damping decrement is approximately

$$b \simeq \frac{\sqrt{\omega_o}}{\sqrt{2}\, c} \left[\sum_j \frac{\omega_{p_j}^2}{\nu_j} \right]^{\frac{1}{2}} .$$

It can be shown from the equation of motion and the expression for the current density that the conductivity σ is given approximately by

$$\sigma \simeq \frac{1}{4\pi} \sum_j \frac{\omega_{p_j}^2}{\nu_j} .$$

Hence,

$$b \simeq \frac{\sqrt{\omega_o}}{c} (2\pi\, \sigma)^{\frac{1}{2}} .$$

We define the depth of penetration, δ, as the reciprocal of b, so that

$$\delta \simeq \frac{c}{\sqrt{2\pi\,\sigma\,\omega_o}} \quad .$$

This is the same as the standard "skin-depth" formula [see, for example, Jackson (1963)] for waves incident on a good conductor. The penetration depth is infinite when $\omega_o = 0$, in agreement with equation (42), which shows no exponential attenuation.

IV. SUMMARY

We have studied the effect of a cold, streaming plasma on the electric and magnetic fields of an oscillating point magnetic dipole. We now summarize our results.

First, we find that the streaming plasma does not distort the magnetic dipole's <u>transverse</u> electromagnetic field for streaming velocities very much less than the velocity of light. However, we also find that a <u>longitudinal</u> electric field appears in the presence of a streaming plasma and that as a result, the Poynting vector is skewed "upstream", against the direction of plasma flow. There is an outward flow of mechanical energy associated with the longitudinal electric field, but the energy flow is negligible for small streaming velocities. Finally, we find that the streaming plasma exerts a "downstream" force on the dipole. Both the force and the longitudinal electric field disappear when the dipole axis is parallel to the direction of flow.

These results are theoretical; they are derived from Maxwell's equations, a dynamical equation and an equation of continuity as applied to a <u>point</u> magnetic dipole in a <u>slowly</u>

streaming plasma. In closing, we should like to propose a simple experiment which one might perform in order to verify (or to discredit) our predictions. The experiment we have in mind is a measurement of the mechanical force which a streaming plasma exerts on an oscillating magnetic dipole. Specifically, one could measure the force required to hold the dipole stationary when the axis of the dipole is normal to the direction of flow. The force should vanish when the two are parallel.

APPENDIX A

Use of the Dynamical Equations

A.1. Derivation of the Equation of Motion

The covariant equation of motion for the ith plasma component is

$$\frac{d}{dt}(\gamma_i \vec{v}_i) = \frac{e_i}{m_i}\left[\vec{E} + \frac{1}{c}(\vec{v}_i \times \vec{B})\right] ,$$

where $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \vec{v}_i \cdot \nabla$ and $\gamma_i \equiv \frac{1}{\sqrt{1 - \frac{v_i^2}{c^2}}}$.

We re-write this equation as

$$\frac{d\vec{v}_i}{dt} = \frac{e_i}{m_i \gamma_i}\left[\vec{E} + \frac{1}{c}(\vec{v}_i \times \vec{B})\right] - \frac{\vec{v}_i}{\gamma_i}\frac{d\gamma_i}{dt} . \qquad \text{(A1)}$$

The last term on the right hand side of (A1) is small since

$$\frac{\vec{v}_i}{\gamma_i}\frac{d\gamma_i}{dt} = -\frac{\vec{v}_i}{2\gamma_i\left(1-\frac{v_i^2}{c^2}\right)^{3/2}}\left(-\frac{1}{c^2}\right)\frac{d}{dt}(\vec{v}\cdot\vec{v})$$

$$= \frac{\gamma_i^2\,\vec{v}_i}{2c^2}\frac{d}{dt}(\vec{v}_i\cdot\vec{v}_i) = \gamma_i^2\,\frac{\vec{v}_i\vec{v}_i}{c^2}\cdot\frac{d\vec{v}_i}{dt}$$

$$\simeq \frac{\vec{v}_i\,\vec{v}_i}{c^2}\cdot\frac{d\vec{v}_i}{dt} + O\left(\frac{v_i^4}{c^4}\right) \quad . \tag{A2}$$

Therefore, the lowest order approximation to the equation of motion is simply

$$\frac{d\vec{v}_i}{dt} = \frac{e_i}{m_i\gamma_i}\left[\vec{E} + \frac{1}{c}(\vec{v}_i \times \vec{B})\right] . \tag{A3}$$

If we substitute (A3) for $\frac{d\vec{v}_i}{dt}$ in (A2) and neglect terms of $O\left(\frac{v_i^4}{c^4}\right)$, the result is

$$\frac{\vec{v}_i}{\gamma_i}\frac{d\gamma_i}{dt} \simeq \frac{\vec{v}_i\,\vec{v}_i}{c^2}\cdot\frac{e_i}{m_i\gamma_i}\left[\vec{E} + \frac{1}{c}(\vec{v}_i \times \vec{B})\right]$$

$$\simeq \frac{e_i}{m_i\gamma_i}\,\frac{\vec{v}_i\vec{v}_i}{c^2}\cdot\vec{E} \quad .$$

Hence, the first order approximation to the equation of motion is

$$\frac{d\vec{v}_i}{dt} = \frac{e_i}{m_i \gamma_i} \left[\vec{E} + \frac{1}{c} (\vec{v}_i \times \vec{B}) - \frac{\vec{v}_i \vec{v}_i}{c^2} \cdot \vec{E} \right] ,$$

correct up to terms of $O\,(v_i^4/c^4)$. This is equation (1b) in the text.

A.2. Derivation of $\vec{v}_i^{(1)}$ and $n_i^{(1)}$ as Functions of $\vec{E}^{(1)}$

<u>a. First order velocity perturbation, $\vec{v}_i^{(1)}$.</u> We solve the linearized equation of motion (2b) for $\vec{v}_i^{(1)}$ as a function of $\vec{E}^{(1)}$. In $(\vec{x},t)$ space equation (2b) is

$$\frac{\partial \vec{v}_i^{(1)}}{\partial t} + \vec{v}_o \cdot \nabla \vec{v}_i^{(1)} = \frac{e_i}{m_i \gamma_o} \left[\vec{E}^{(1)} + \frac{1}{c} (\vec{v}_o \times \vec{B}^{(1)}) - \frac{\vec{v}_o \vec{v}_o}{c^2} \cdot \vec{E}^{(1)} \right]$$

Fourier transformation ($\nabla \rightarrow i\vec{k}$; $\frac{\partial}{\partial t} \rightarrow -i\omega$) gives

$$-i\omega\, \vec{v}_i^{(1)} + i\vec{v}_i^{(1)} (\vec{k} \cdot \vec{v}_o) = \frac{e_i}{m_i \gamma_o} \left[\vec{E}^{(1)} + \frac{1}{c} (\vec{v}_o \times \vec{B}^{(1)}) - \frac{\vec{v}_o \vec{v}_o}{c^2} \cdot \vec{E}^{(1)} \right]$$

By Faraday's Law $\nabla \times \vec{E}^{(1)} = -\frac{1}{c}\frac{\partial \vec{B}^{(1)}}{\partial t}$, or $\vec{B}^{(1)} = \left[\frac{c}{\omega}\ \vec{k} \times \vec{E}^{(1)}\right]$ in transform space. Hence,

$$i\,\vec{v}_i^{(1)}[(\vec{k}\cdot\vec{v}_o) - \omega] = \frac{e_i}{m_i\gamma_o}\left\{\vec{E}^{(1)} + \frac{\vec{v}_o}{\omega} \times (\vec{k} \times \vec{E}^{(1)}) - \frac{\vec{v}_o\vec{v}_o}{c^2}\cdot\vec{E}^{(1)}\right\}$$

Using the identity $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$, we find that

$$i\vec{v}_i^{(1)}[(\vec{k}\cdot\vec{v}_o) - \omega] = \frac{e_i}{m_i\gamma_o}\left\{\vec{E}^{(1)} + \frac{1}{\omega}\left[\vec{k}(\vec{v}_o\cdot\vec{E}^{(1)}) - (\vec{v}_o\cdot\vec{k})\vec{E}^{(1)}\right] - \frac{\vec{v}_o\vec{v}_o}{c^2}\cdot\vec{E}^{(1)}\right\}$$

or

$$-i\vec{v}_i^{(1)}\left(1 - \frac{\vec{k}\cdot\vec{v}_o}{\omega}\right)\omega = \frac{e_i}{m_i\gamma_o}\left\{\left(1 - \frac{\vec{k}\cdot\vec{v}_o}{\omega}\right)\vec{E}^{(1)} + \frac{1}{\omega}(\vec{k}\ \vec{v}_o)\cdot\vec{E}^{(1)} - \frac{\vec{v}_o\vec{v}_o}{c^2}\cdot\vec{E}^{(1)}\right\}.$$

The solution for $\vec{v}_i^{(1)}$ is

$$\vec{v}_i^{(1)} = -\frac{e_i}{m_i \gamma_o (i\omega)} \left\{ \vec{\vec{I}} + \frac{\vec{k}\,\vec{v}_o}{(\omega - \vec{k}\cdot\vec{v}_o)} - \frac{\omega\;\vec{v}_o\vec{v}_o}{c^2(\omega - \vec{k}\cdot\vec{v}_o)} \right\} \cdot \vec{E}^{(1)}$$

which is equation (5a) in the text.

b. First Order Number Density Perturbation $n_i^{(1)}$. The linearized equation of continuity, (2a), is

$$\frac{\partial n_i^{(1)}}{\partial t} + n_{o_i}\,\nabla\cdot\vec{v}_i^{(1)} + (\nabla n_i^{(1)})\cdot\vec{v}_o = 0.$$

After Fourier transformation, the equation becomes

$$-i\,(\omega - \vec{k}\cdot\vec{v}_o)\,n_i^{(1)} + i\,n_{o_i}(\vec{k}\cdot\vec{v}_i^{(1)}) = 0$$

Therefore,

$$n_i^{(1)} = \frac{n_{o_i}(\vec{k}\cdot\vec{v}_i^{(1)})}{(\omega - \vec{k}\cdot\vec{v}_o)} .$$

We substitute expression (5a) for $\vec{v}_i^{(1)}$. The result is

$$n_i^{(1)} = - \frac{n_{o_i} e_i}{m_i \gamma_o (i\omega)} \quad \frac{\vec{k}}{(\omega - \vec{k}\cdot\vec{v}_o)} \cdot \left\{ \vec{\vec{1}} + \frac{\vec{k}\,\vec{v}_o}{(\omega - \vec{k}\cdot\vec{v}_o)} - \frac{\omega\,\vec{v}_o\vec{v}_o}{c^2(\omega - \vec{k}\cdot\vec{v}_o)} \right\} \cdot \vec{E}^{(1)},$$

which is equation (5b) in the text.

A.3. The Plasma Current Density $\vec{j}_p^{\,(1)}(\vec{k},\omega)$ as a Function of $\vec{E}^{(1)}(\vec{k},\omega)$

By equation (3) in the text (without $\vec{j}_s$),

$$\vec{j}_p^{\,(1)} = \sum_i e_i \left[n_{o_i} \vec{v}_i^{\,(1)} + \vec{v}_o\, n_i^{(1)} \right] .$$

We substitute for $\vec{v}_i^{\,(1)}$ and $n_i^{(1)}$, using the results of A.2:

$$\vec{j}_p^{\,(1)} = - \sum_i \frac{n_{o_i} e_i^2}{m_i (i\omega)\gamma_o} \left\{ \left[\vec{\vec{1}} + \frac{\vec{k}\,\vec{v}_o}{(\omega - \vec{k}\cdot\vec{v}_o)} - \frac{\omega\,\vec{v}_o\vec{v}_o}{c^2(\omega - \vec{k}\cdot\vec{v}_o)} \right] \right.$$

$$\left. + \frac{\vec{v}_o\vec{k}}{(\omega - \vec{k}\cdot\vec{v}_o)} \cdot \left[\vec{\vec{1}} + \frac{\vec{k}\,\vec{v}_o}{(\omega\ \vec{k}\cdot\vec{v}_o)} - \frac{\omega\,\vec{v}_o\vec{v}_o}{c^2(\omega - \vec{k}\cdot\vec{v}_o)} \right] \right\} \cdot \quad \vec{E}^{(1)}$$

Since $\omega_{p_i}^2 \equiv 4\pi \dfrac{n_{o_i} e_i^2}{m_i}$ (plasma frequency),

$$\vec{j}_p^{\,(1)} = -\frac{1}{4\pi(i\omega)\gamma_o}\left(\sum_i \omega_{p_i}^2\right)\left\{\vec{\vec{1}} + \frac{1}{(\omega-\vec{k}\cdot\vec{v}_o)}[\vec{k}\,\vec{v}_o + \vec{v}_o\,\vec{k}]\right.$$

$$\left. + \frac{1}{(\omega-\vec{k}\cdot\vec{v}_o)^2}\left[k^2 - \frac{\omega(\omega-\vec{k}\cdot\vec{v}_o)}{c^2} - \frac{\omega(\vec{k}\cdot\vec{v}_o)}{c^2}\right]\vec{v}_o\vec{v}_o\right\}\cdot\vec{E}^{(1)}$$

or

$$\vec{j}_p^{\,(1)} = -\frac{1}{4\pi(i\omega)\gamma_o}\left(\sum_i \omega_{p_i}^2\right)\left\{\vec{\vec{1}} + \frac{1}{(\omega-\vec{k}\cdot\vec{v}_o)}[\vec{k}\,\vec{v}_o + \vec{v}_o\,\vec{k}]\right.$$

$$\left. + \frac{(k^2 - \omega^2/c^2)\,\vec{v}_o\vec{v}_o}{(\omega-\vec{k}\cdot\vec{v}_o)^2}\right\}\cdot\vec{E}^{(1)} .$$

The expression above may be substituted for $\vec{j}_p^{\,(1)}$ in the wave equation; the result is equation (6) in the text.

APPENDIX B

Use of Maxwell's Equations

B.1. The Wave Equation

We derive a wave equation from the two curl equations of Maxwell, (1c) and (1d) in the text. These equations are $\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$ and $\nabla \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{j}$. Their Fourier transforms ($\nabla \rightarrow ik$ and $\frac{\partial}{\partial t} \rightarrow -i\omega$) are $i\vec{k} \times \vec{E} = \frac{i\omega}{c}\vec{B}$ and $ik \times \vec{B} = -\frac{i\omega}{c}\vec{E} + \frac{4\pi}{c}\vec{j}$, respectively. We take the curl of the first equation ($i\vec{k} \times [\quad]$ in $\vec{k},\omega$ space) and substitute for $i\vec{k} \times \vec{B}$ in the second. The result is

$$\frac{c}{i\omega}[i\vec{k} \times (i\vec{k} \times \vec{E})] = -\frac{i\omega}{c}\vec{E} + \frac{4\pi}{c}\vec{j}$$

or

$$(i\vec{k} \cdot \vec{E})\, i\vec{k} + k^2\vec{E} = \frac{\omega^2}{c^2}\vec{E} + \frac{4\pi i\omega}{c^2}\vec{j}$$

We re-write this equation as

$$[\ \vec{\vec{1}}\ (\omega^2 - c^2k^2) \ + \ c^2\ \overleftrightarrow{kk}\ ] \cdot \ \vec{E}^{(1)} \ = \ -\ 4\pi\ i\omega\ \vec{j}\ .$$

It is now equivalent to equation (4) in the text, without the expanded current density.

B.2. The Two Scalar Maxwell Equations

We remark here that it is easily shown from the vector Maxwell equations and the equation of continuity that

$$\frac{\partial}{\partial t}(\ \nabla \cdot \vec{B}\ ) = \ 0$$

$$\frac{\partial}{\partial t}\ (\nabla \cdot \vec{E} \ - \ 4\pi\ \rho) = 0\ ,$$

so that if the scalar equations are satisfied initially, they will be satisfied for all time, as a consequence of Eqs. (1a), (1c) and (1d) alone. We can determine the charge density either from Poisson's equation or from Eq. (6) when static fields are absent. The latter procedure is more convenient.

APPENDIX C

Derivation of $\vec{E}_{\perp}^{(1)}(\vec{k},\omega)$ and $\vec{E}_{\parallel}^{(1)}(\vec{k},\omega)$ as Functions of $\vec{j}_{source}(\vec{k},\omega)$ and $\vec{v}_o$

In this Appendix we carry out the program outlined on pages 10 and 11 of the text. We begin with equation (6):

$$\left[\,\vec{\vec{I}}\,(\omega^2 - c^2k^2) + c^2\vec{k}\vec{k}\,\right]\cdot\vec{E}^{(1)} - \frac{\left(\sum_i \omega_{p_i}^{\,2}\right)}{\gamma_o}\left\{\vec{\vec{I}} + \frac{1}{(\omega-\vec{k}\cdot\vec{v}_o)}\left[\,\vec{k}\,\vec{v}_o + \vec{v}_o\,\vec{k}\,\right] + \frac{(k^2 - \omega^2/c^2)\,\vec{v}_o\vec{v}_o}{(\omega-\vec{k}\cdot\vec{v}_o)^2}\right\}\cdot\vec{E}^{(1)} = -\,4\pi\, i\omega\,\vec{j}_s$$

This is the wave equation, with the expression derived in Appendix B substituted for the plasma current density $\vec{j}_p$. For conciseness, we write this equation as $\vec{\vec{R}}\cdot\vec{E}^{(1)} = -\,4\pi\, i\omega\,\vec{j}_s$, where $\vec{\vec{R}}$ may be termed the "resistivity" tensor. With the symbolism

$$A = \frac{1}{(\omega - \vec{k}\cdot\vec{v}_o)} \quad \frac{\sum\limits_i \omega_{p_i}^{\,2}}{\gamma_o}$$

$$B = \frac{1}{(\omega - \vec{k}\cdot\vec{v}_o)^2} \quad \frac{\sum\limits_i \omega_{p_i}^{\,2}}{\gamma_o} \left(k^2 - \frac{\omega^2}{c^2} \right)$$

$$H = \frac{\sum\limits_i \omega_{p_i}^{\,2}}{\gamma_o} \quad ,$$

$$\vec{\vec{R}} = \left\{ \left[\vec{\vec{I}}(\omega^2 - c^2k^2 - H) + c^2\, \vec{k}\vec{k} \right] - A\, [\vec{k}\, \vec{v}_o + \vec{v}_o \vec{k}] - B\, \vec{v}_o \vec{v}_o \right\} .$$

C.1. Resolution of the Wave Equation into Transverse and Longitudinal Components

First, we write $\vec{\vec{R}} \cdot \vec{E}^{(1)} = -4\pi i\omega\, \vec{j}_s$ explicitly in terms of $\vec{E}_{\perp}^{(1)}$ and $\vec{E}_{\parallel}^{(1)}$, where "$\perp$" and "$\parallel$" mean perpendicular and parallel to $\vec{k}$.

$$\vec{R} \cdot \vec{E}^{(1)} = \Bigg\{ (\omega^2 - c^2k^2 - H)\, \vec{E}_\perp^{(1)} - A\vec{k}\, (\vec{v}_o \cdot \vec{E}_\perp^{(1)})$$

$$- B\, \vec{v}_o\, (\vec{v}_o \cdot \vec{E}_\perp^{(1)}) + (\omega^2 - H)\, \vec{E}_\parallel^{(1)}$$

$$- A\, \vec{k}\, (\vec{v}_o \cdot \vec{E}_\parallel^{(1)}) - A\, \vec{v}_o\, (\vec{k} \cdot E_\parallel^{(1)})$$

$$- B\, \vec{v}_o\, (\vec{v}_o \cdot \vec{E}_\parallel^{(1)}) \Bigg\} = -4\pi\, i\omega\, \vec{j}_s$$

The equation above is a partial resolution of (6) into components which are perpendicular and parallel to $\vec{k}$. Similarly dividing $\vec{v}_o$, $\vec{v}_o = \vec{v}_{o_\perp} + \vec{v}_{o_\parallel}$, we complete the resolution as follows:

<u>Transverse Component</u>:

$$[\omega^2 - c^2k^2 - H]\, \vec{E}_\perp^{(1)} - B\, [\vec{v}_o \cdot \vec{E}_\perp^{(1)}]\, \vec{v}_{o_\perp}$$

$$-A\, [\vec{k} \cdot \vec{E}_\parallel^{(1)}]\, \vec{v}_{o_\perp} - B\, [\vec{v}_o \cdot \vec{E}_\parallel^{(1)}]\, \vec{v}_{o_\perp} = -4\pi\, i\omega\, j_{s_\perp}$$

Longitudinal Component

$$[\omega^2 - H]\,\vec{E}_{\parallel}^{(1)} - A\,[\vec{v}_o \cdot \vec{E}_{\perp}^{(1)}]\,\vec{k} - A\,[\vec{v}_o \cdot \vec{E}_{\parallel}^{(1)}]\,\vec{k}$$

$$- A\,[\vec{k} \cdot \vec{E}_{\parallel}^{(1)}]\,\vec{v}_{o_{\parallel}} - B\,[\vec{v}_o \cdot \vec{E}_{\perp}^{(1)}]\,\vec{v}_{o_{\parallel}} - B[\vec{v}_o \cdot \vec{E}_{\parallel}^{(1)}]\vec{v}_{o_{\parallel}}$$

$$= -4\pi\, i\omega\, j_{s_{\parallel}}$$

C.2. Solution of the Transverse Component for $\vec{v}_o \cdot \vec{E}_{\perp}^{(1)}$

Next, we dot the velocity vector $\vec{v}_o$ into the transverse component and solve for $\vec{v}_o \cdot \vec{E}_{\perp}^{(1)}$.

$$[\omega^2 - c^2k^2 - H]\,[\vec{v}_o \cdot \vec{E}_{\perp}^{(1)}] - B\,[\vec{v}_o \cdot \vec{E}_{\perp}^{(1)}]\,v_{o_{\perp}}^{\,2}$$

$$= A[k\,E_{\parallel}^{(1)}]\,v_{o_{\perp}}^{\,2} + B[\vec{v}_o \cdot \vec{E}_{\parallel}^{(1)}]\,v_{o_{\perp}}^{\,2} - 4\pi\, i\omega\,(\vec{j}_{s_{\perp}} \cdot \vec{v}_o)$$

or

$$D[\vec{v}_o \cdot \vec{E}_{\perp}^{(1)}] = F\,[E_{\parallel}^{(1)}\,v_{o_{\perp}}^{\,2}] - 4\pi\, i\omega\,[\vec{j}_{s_{\perp}} \cdot \vec{v}_o]$$

where $D = \omega^2 - c^2k^2 - H - B\, v_{o_\perp}^{\;2}$

and $F = Ak + Bv_{o_\parallel}$.

C.3. Solution of the Longitudinal Component for $E_\parallel^{(1)}$

We write a scalar equation for $E_\parallel^{(1)}$, since $\vec{E}_\parallel^{(1)} \parallel \vec{k} \parallel \vec{v}_{o_\parallel} \parallel \vec{j}_{s_\parallel}$.

$$[\omega^2 - H]\, E_\parallel^{(1)} - A[v_{o_\parallel}\, k]\, E_\parallel^{(1)} - A[v_{o_\parallel}\, k]\, E_\parallel^{(1)}$$

$$-B\, v_{o_\parallel}^{\;2}\, E_\parallel^{(1)} = A\, k[\vec{v}_o \cdot \vec{E}_\perp^{(1)}] + B\, v_{o_\parallel}\, [\vec{v}_o \cdot \vec{E}_\perp^{(1)}]$$

$$- 4\pi\, i\omega\, j_{s_\parallel}$$

or

$$GE_\parallel^{(1)} = F(\vec{v}_o \cdot \vec{E}_\perp^{(1)}) - 4\pi\, i\omega\, j_{s_\parallel}$$

where $G = [\,\omega^2 - H - 2A\, k\, v_{o_\parallel} - B\, v_{o_\parallel}^{\;2}]$

C.4 Simultaneous Solution for $E_{\parallel}^{(1)}$, $\vec{v} \cdot \vec{E}_{\perp}^{(1)}$ and $\vec{E}_{\perp}^{(1)}$

a. Solution for $E_{\parallel}^{(1)}$. We now have two equations in $E_{\parallel}^{(1)}$ and $\vec{v}_o \cdot \vec{E}_{\perp}^{(1)}$. The solution for $E_{\parallel}^{(1)}$ proceeds as follows:

$$E_{\parallel}^{(1)} = \frac{F}{G}\left[\left(\frac{Fv_{o_\perp}{}^2}{D}\right) E_{\parallel}^{(1)} - \frac{4\pi\, i\omega}{D}\ (\vec{j}_{s_\perp} \cdot \vec{v}_o)\right] - \frac{4\pi\, i\omega}{G}\ (j_{s_\parallel})$$

$$E_{\parallel}^{(1)}\left[1 - \frac{F^2}{GD}\ v_{o_\perp}{}^2\right] = -\frac{4\pi\, i\omega\, F}{GD}\ (\vec{j}_{s_\perp} \cdot \vec{v}_o) - \frac{4\pi\, i\omega}{G}\ (j_{s_\parallel})$$

$$E_{\parallel}^{(1)} = -\frac{4\pi\, i\omega}{[GD - F^2 v_{o_\perp}{}^2]}\left[F(\vec{j}_{s_\perp} \cdot \vec{v}_o) + D(j_{s_\parallel})\right]$$

b. Solution for $\vec{v}_o \cdot \vec{E}_{\perp}^{(1)}$. Next, we use the expression for $E_{\parallel}^{(1)}$ to solve for $\vec{v}_o \cdot \vec{E}_{\perp}^{(1)}$:

$$D[\vec{v}_o \cdot \vec{E}_{\perp}^{(1)}] = F[E_{\parallel}^{(1)}\ v_{o_\perp}{}^2] - 4\pi\, i\omega\, [\vec{j}_{s_\perp} \cdot \vec{v}_o]$$

$$= -\frac{4\pi\, i\omega\, D}{[GD - F^2 v_{o_\perp}{}^2]}\left[G(\vec{j}_{s_\perp} \cdot \vec{v}_o) + F\ (j_{s_\parallel}\ v_{o_\perp}{}^2)\right]$$

$$[\vec{v}_o \cdot \vec{E}_{\perp}^{(1)}] = -\frac{4\pi\, i\omega}{[GD - F^2 v_{o_\perp}{}^2]}\left[G(\vec{j}_{s_\perp} \cdot \vec{v}_o) + F(j_{s_\parallel}\ v_{o_\perp}{}^2)\right]$$

c. Solution for $\vec{E}_\perp^{(1)}$. Finally, we return to the transverse component and solve for $\vec{E}_\perp^{(1)}$, remembering that $F = Ak + Bv_{o_\parallel}$:

$$[\omega^2 - c^2k^2 - H]\,\vec{E}_\perp^{(1)} - B[\vec{v}_o \cdot \vec{E}_\perp^{(1)}]\,\vec{v}_{o_\perp} - F\,E_\parallel^{(1)}\,\vec{v}_{o_\perp} = -4\pi\,i\omega\,\vec{j}_{s_\perp}$$

$$[\omega^2 - c^2k^2 - H]\,\vec{E}_\perp^{(1)} + \frac{4\pi\,i\omega\,B}{[GD - F^2 v_{o_\perp}^{\;2}]}\left[G(\vec{j}_{s_\perp}\cdot\vec{v}_o) + F(j_{s_\parallel}\,v_{o_\perp}^{\;2})\right]\vec{v}_{o_\perp}$$

$$+\;\frac{4\pi\,i\omega\,F}{[GD - F^2 v_{o_\perp}^{\;2}]}\left[F(\vec{j}_{s_\perp}\cdot\vec{v}_o) + D(j_{s_\parallel})\right]\vec{v}_{o_\perp} \;=\; -\,4\pi\,i\omega\,\vec{j}_{s_\perp}$$

$$\vec{E}_\perp^{(1)} = -\,\frac{4\pi\,i\omega}{[\omega^2 - c^2k^2 - H]}\left\{\vec{j}_{s_\perp} + \frac{[(BG+F^2)(\vec{j}_{s_\perp}\cdot\vec{v}_o) + F(D+Bv_{o_\perp}^{\;2})(j_{s_\parallel})]\,\vec{v}_{o_\perp}}{[GD - F^2\,v_{o_\perp}^{\;2}]}\right\}$$

C.5. Expressions for $\vec{E}_\perp^{(1)}(\vec{k},\omega)$ and $\vec{E}_\parallel^{(1)}(\vec{k},\omega)$ when $j_{s_\parallel} = 0$

For $j_{s_\parallel} = 0$, the solutions above for $\vec{E}_\perp^{(1)}$ and $\vec{E}_\parallel^{(1)}$ become

$$\vec{E}_{\perp}^{(1)} = - \frac{4\pi\, i\omega}{[\omega^2 - c^2k^2 - H]} \left\{ \vec{j}_{s_\perp} + \frac{[(BG + F^2)\,(\vec{j}_{s_\perp} \cdot \vec{v}_o)]\, \vec{v}_{o_\perp}}{[GD - F^2\, v_{o_\perp}^{\;2}]} \right\}$$

$$E_{\parallel}^{(1)} = - \frac{4\pi\, i\omega\, F\, (\vec{j}_{s_\perp} \cdot \vec{v}_o)}{[GD - F^2\, v_{o_\perp}^{\;2}]}$$

We may make these expressions more meaningful by writing F, $BG + F^2$ and $GD - F^2\, v_{o_\perp}^{\;2}$ explicitly in terms of ω, k, ω_p and $\vec{v}_o$.

<u>The factor F in the numerator of $E_{\parallel}^{(1)}$</u>:

$$F = Ak + Bv_{o_\parallel}$$

$$= \frac{\left(\sum_i \omega_{p_i}^{\;2}\right) k}{\gamma_o(\omega - \vec{k}\cdot\vec{v}_o)} + \frac{\left(\sum_i \omega_{p_i}^{\;2}\right)\left(k^2 - \frac{\omega^2}{c^2}\right) v_{o_\parallel}}{\gamma_o \quad (\omega - \vec{k}\cdot\vec{v}_o)^2}$$

$$F = \frac{\left(\sum_i \omega_{p_i}^{\;2}\right) \omega}{\gamma_o(\omega - \vec{k}\cdot\vec{v}_o)^2} \left[k - \frac{\omega}{c^2}\, v_{o_\parallel} \right]$$

<u>The symbol B</u>:

$$B = \frac{\sum_i \omega_{p_i}^{\;2}}{\gamma_o} \qquad \frac{(k^2 - \omega^2/c^2)}{(\omega - \vec{k}\cdot\vec{v}_o)^2}$$

The symbol G:

$$G = \omega^2 - H - 2Akv_{o_\parallel} - Bv_{o_\parallel}^{\;2}$$

$$= \omega^2 - \frac{\left(\sum\limits_i \omega_{p_i}^{\;2}\right)}{\gamma_o} - \frac{2\left(\sum\limits_i \omega_{p_i}^{\;2}\right)\left(\vec{k}\cdot\vec{v}_o\right)}{\gamma_o(\omega - \vec{k}\cdot\vec{v}_o)} - \frac{\left(\sum\limits_i \omega_{p_i}^{\;2}\right)\left(k^2 - \frac{\omega^2}{c^2}\right) v_{o_\parallel}^{\;2}}{\gamma_o \quad (\omega - \vec{k}\cdot\vec{v}_o)^2}$$

$$G = \omega^2 \left[1 - \frac{\left(\sum\limits_i \omega_{p_i}^{\;2}\right)\left(1 - \frac{v_{o_\parallel}^{\;2}}{c^2}\right)}{\gamma_o \quad (\omega - \vec{k}\cdot\vec{v}_o)^2}\right]$$

The factor $BG + F^2$ in the numerator of $\vec{E}_\perp^{(1)}$

$$BG + F^2 = \frac{\left(\sum\limits_i \omega_{p_i}^{\;2}\right) \omega^2\left(k^2 - \frac{\omega^2}{c^2}\right)}{\gamma_o \quad (\omega - \vec{k}\cdot\vec{v}_o)^2} - \left(\frac{\sum\limits_i \omega_{p_i}^{\;2}}{\gamma_o}\right)^2 \frac{\omega^2\left(k^2 - \frac{\omega^2}{c^2}\right)\left(1 - \frac{v_{o_\parallel}^{\;2}}{c^2}\right)}{(\omega - \vec{k}\cdot\vec{v}_o)^4}$$

$$+ \left(\frac{\sum\limits_i \omega_{p_i}^{\;2}}{\gamma_o}\right)^2 \frac{\omega^2\left(k - \frac{\omega}{c^2}\, v_{o_\parallel}\right)^2}{(\omega - \vec{k}\cdot\vec{v}_o)^4}$$

$$BG + F^2 = -\left(\frac{\sum_i \omega_{p_i}^{\ 2}}{\gamma_o}\right) \frac{\omega^2}{c^2(\omega - \vec{k}\cdot\vec{v}_o)^2} \left[\omega^2 - c^2k^2 - \frac{\sum_i \omega_{p_i}^{\ 2}}{\gamma_o}\right]$$

The symbol D:

$$D = \omega^2 - c^2k^2 - H - Bv_{o_\perp}^{\ 2}$$

$$= \omega^2 - c^2k^2 - \frac{\sum_i \omega_{p_i}^{\ 2}}{\gamma_o} - \frac{\sum_i \omega_{p_i}^{\ 2} \; (c^2k^2 - \omega^2)\, v_{o_\perp}^{\ 2}}{\gamma_o \; (\omega - \vec{k}\cdot\vec{v}_o)^2 \, c^2}$$

The Factor $GD-F^2 v_{o_\perp}^{\ 2}$ in the denominators of $\vec{E}_\parallel^{(1)}$ and $\vec{E}_\perp^{(1)}$

$$GD - F^2 v_{o_\perp}^{\ 2} = \omega^2 \left[1 - \left(\frac{\sum_i \omega_{p_i}^{\ 2}}{\gamma_o}\right) \frac{\left(1 - \frac{v_{o_\parallel}^{\ 2}}{c^2}\right)}{(\omega - \vec{k}\cdot\vec{v}_o)^2}\right]$$

$$\cdot \left[\omega^2 - c^2k^2 - \frac{\left(\sum_i \omega_{p_i}^{\ 2}\right)}{\gamma_o}\right]$$

$$- \omega^2 \left(\frac{\sum_i \omega_{p_i}^{\ 2}}{\gamma_o}\right) \left[\frac{(c^2k^2 - \omega^2)\; v_{o_\perp}^{\ 2}}{(\omega - \vec{k}\cdot\vec{v}_o)^2 \; c^2} + \frac{\left(\sum_i \omega_{p_i}^{\ 2}\right) v_{o_\perp}^{\ 2}}{\gamma_o \; (\omega - \vec{k}\cdot\vec{v}_o)^2 \; c^2}\right]$$

$$= \omega^2 \left\{ \omega^2 - c^2k^2 - \frac{\sum\limits_i \omega_{p_i}^{\ 2}}{\gamma_o} \right\} \left\{ 1 - \frac{\sum\limits_i \omega_{p_i}^{\ 2}}{\gamma_o} \left[\frac{1 - \frac{v_{o_\parallel}^{\ 2}}{c^2} - \frac{v_{o_\perp}^{\ 2}}{c^2}}{(\omega - \vec{k}\cdot\vec{v}_o)^2} \right] \right\}$$

$$= \omega^2 \left[\omega^2 - c^2k^2 - \frac{\sum\limits_i \omega_{p_i}^{\ 2}}{\gamma_o} \right] \left[1 - \frac{\sum\limits_i \omega_{p_i}^{\ 2}}{\gamma_o^{\ 3}\,(\omega - \vec{k}\cdot\vec{v}_o)^2} \right]$$

Using the preceeding results, we finally obtain equations (7) and (8) of the text:

$$\vec{E}_\perp^{(1)}(\vec{k},\omega) = - \frac{4\pi\, i\omega}{\left[\omega^2 - c^2k^2 - \frac{\sum\limits_i \omega_{p_i}^{\ 2}}{\gamma_o} \right]}$$

$$\left\{ \vec{j}_{s_\perp} - \frac{\left(\sum\limits_i \omega_{p_i}^{\ 2}\right)\left(\vec{j}_{s_\perp} \cdot \vec{v}_o\right)\ \vec{v}_{o_\perp}}{\gamma_o c^2 \left[(\omega - \vec{k}\cdot\vec{v}_o)^2 - \frac{\sum\limits_i \omega_{p_i}^{\ 2}}{\gamma_o^{\ 3}} \right]} \right\}$$

which is equation (7), and

$$\vec{E}_{\parallel}^{(1)}(\vec{k},\omega) = \frac{-4\pi i \left(\sum_i \omega_{p_i}^2\right)\left[1 - \frac{\omega}{c^2}\left(\frac{\vec{k}\cdot\vec{v}_o}{k^2}\right)\right]\left[\vec{j}_{s_\perp}\cdot\vec{v}_o\right]\vec{k}}{\gamma_o\left[\omega^2 - c^2k^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o}\right]\left[(\omega - \vec{k}\cdot\vec{v}_o)^2 - \frac{\sum_i \omega_{p_i}^2}{\gamma_o^3}\right]}$$

which is equation (8).

APPENDIX D

Derivation of $\vec{J}_{source_\perp}(\vec{k},\omega)$ for Magnetic Dipole

Consider a small current loop of radius a_o, with axis parallel to the z axis of an (x,y,z) coordinate system (Figure 1). The current density is (in terms of the unit basis vectors $\hat{e}_r$, $\hat{e}_\varphi$, $\hat{e}_z$):

$$\vec{J}_s = J_o \, \hat{e}_\varphi \, \delta \, (r - a_o) \delta \, (z) \, ,$$

where J_o is the loop current and r, φ, and z are cylindrical coordinates. This expression is equivalent to $\vec{J}_s = J_o(\hat{e}_y \cos \varphi - \hat{e}_x \sin \varphi) \, \delta \, (r - a_o) \delta \, (z)$, in terms of the fixed Cartesian unit vectors $\hat{e}_x$ and $\hat{e}_y$.

Transform in Space

The spatial transform of the preceding equation is

$$\vec{J}(\vec{k}) = \frac{J_o}{(2\pi)^3} \int r dr d\varphi dz e^{-i(\vec{k}\cdot\vec{x})} \, (\hat{e}_y \cos\varphi - \hat{e}_x \sin\varphi) \, \delta \, (r-a_o) \delta \, (z)$$

$$= \frac{J_o}{(2\pi)^3} \int r dr d\varphi dz e^{-i(k_x \cos\varphi + k_y \sin\varphi)r - ik_z z} (\hat{e}_y \cos\varphi - \hat{e}_x \sin\varphi)$$

$$\delta(r - a_o) \, \delta \, (z).$$

The r and z integrations are easy:

$$\int dz \int r dr \; e^{-i(k_x \cos\varphi + k_y \sin\varphi)r - ik_z z} \; \delta(r - a_o)\,\delta(z)$$

$$= a_o e^{-i(k_x \cos\varphi + k_y \sin\varphi)\, a_o} \;.$$

For the φ integration we expand the exponential:

$$\vec{j}(\vec{k}) = \frac{\left(J_o a_o\right)}{(2\pi)^3} \int_o^{2\pi} d\varphi \,[1 - i(k_x \cos\varphi + k_y \sin\varphi)a_o + O(a_o^2)\ldots.]$$

$$[\hat{e}_y \cos\varphi - \hat{e}_x \sin\varphi]$$

$$= -i \frac{\left(J_o a_o^2\right)}{(2\pi)^3} \int_o^{2\pi} d\varphi \,[\hat{e}_y \,(k_x \cos^2 \varphi) - \hat{e}_x \,(k_y \sin^2 \varphi)] + O\left(J_o a_o^4\right)$$

$$\vec{j}(\vec{k}) = -i \frac{\pi\left(J_o a_o^2\right)}{(8\pi^3)} \quad [k_x \hat{e}_y - k_y \hat{e}_x] \;+ O\left(J_o a_o^4\right) .$$

By definition, for $\hat{e}_z$ normal to the current loop,

$$\left|\vec{\mu}\right| = \lim_{\substack{a_o \to 0 \\ J_o \to \infty \\ J_o a_o^2 \to \text{constant}}} \frac{\pi}{c}\,(J_o a_o^2) = \mu_z$$

Hence, in the limit of a point dipole,

$$\vec{j}(\vec{k}) = i\left(\frac{\mu_z c}{8\pi^3}\right)\left[(k_y\,\hat{e}_x) - (k_x\,\hat{e}_y)\right]$$

$$= \frac{ic}{8\pi^3}\left[(k_y\,\mu_z)\,\hat{e}_x - (k_x\,\mu_z)\,\hat{e}_y\right]$$

$$= \frac{ic}{8\pi^3}\left[(\vec{k} \times \vec{\mu})_x\,\hat{e}_x + (\vec{k} \times \vec{\mu})_y\,\hat{e}_y\right]$$

$$\vec{j}(\vec{k}) = \frac{ic}{8\pi^3}\left[\vec{k} \times \vec{\mu}\right],$$

where $\vec{\mu}$ still has arbitrary time dependence.

Transform in Time

The temporal transform of $\vec{\mu}(t)$ is

$$\vec{\mu}(\omega) = \frac{1}{2\pi} \int dt e^{i\omega t} \vec{\mu}(t) .$$

We specify a sinusoidal time dependence for the current loop:

$$\vec{\mu}(t) = \vec{\mu}_o \cos \omega_o t ,$$

where ω_o is the external driving frequency. Consequently,

$$\vec{\mu}(\omega) = \left\{ \frac{1}{2\pi} \int dt\, e^{i\omega t} \left[\frac{e^{i\omega_o t} + e^{-i\omega_o t}}{2} \right] \right\} \vec{\mu}_o$$

$$= \left\{ \frac{2\pi}{4\pi} \left[\delta(\omega - \omega_o) + \delta(\omega + \omega_o) \right] \right\} \vec{\mu}_o .$$

$\vec{j}(\vec{k},\omega)$ for Magnetic Dipole

Therefore, the Fourier transform of the current loop is

$$\vec{j}(\vec{k},\omega) = \frac{ic}{16\pi^3} [\vec{k} \times \vec{\mu}_o] [\delta (\omega - \omega_o) + \delta (\omega + \omega_o)].$$

The current density for the magnetic dipole is purely transverse since

$$\vec{k} \cdot \vec{j}(\vec{k},\omega) = 0 .$$

APPENDIX E

Derivation of $\vec{E}_{\perp}^{(1)}(\vec{x},t)$ and $\vec{E}_{\parallel}^{(1)}(\vec{x},t)$: Separation into Parial Fractions

We wish to find the inverse transforms of equations (7) and (8) of the text, using equation (9) for $\vec{j}(\vec{k},\omega)$. We facilitate their integration by separating

$$\left\{\left[\omega^2 - c^2k^2 - \frac{\sum_i \omega_{p_i}^{\ 2}}{\gamma_o}\right]\left[(\omega - \vec{k}\cdot\vec{v}_o)^2 - \frac{\sum_i \omega_{p_i}^{\ 2}}{\gamma_o^{\ 3}}\right]\right\}^{-1}$$

into partial fractions. Let

$$\frac{1}{\left[\omega^2 - c^2k^2 - \frac{\sum_i \omega_{p_i}^{\ 2}}{\gamma_o}\right]\left[(\omega - \vec{k}\cdot\vec{v}_o)^2 - \frac{\sum_i \omega_{p_i}^{\ 2}}{\gamma_o^{\ 3}}\right]} =$$

$$\frac{Mk + N}{\left[\omega^2 - c^2k^2 - \frac{\sum_i \omega_{p_i}^{\ 2}}{\gamma_o}\right]} + \frac{Pk + Q}{\left[(\omega - \vec{k}\cdot\vec{v}_o)^2 - \frac{\sum_i \omega_{p_i}^{\ 2}}{\gamma_o^{\ 3}}\right]}$$

We will solve for the unknown coefficients M, N, P, and Q.

Polynomial Formation

The equation above is equivalent to

$$(Mk + N)\,[\Omega_3^{\,2} - 2\omega(\vec{k}\cdot\vec{v}_o) + (\vec{k}\cdot\vec{v}_o)^2]$$

$$+ (Pk + Q)\,[\Omega_1^{\,2} - c^2k^2] = 1$$

$$\text{where } \Omega_1^{\,2} = \left(\omega^2 - \frac{\sum\limits_i \omega_{p_i}^{\,2}}{\gamma_o}\right)$$

$$\text{and } \Omega_3^{\,2} = \left(\omega^2 - \frac{\sum\limits_i \omega_{p_i}^{\,2}}{\gamma_o^{\,3}}\right).$$

Expansion of this expression gives a polynomial in k:

$$(M\Omega_3^{\,2})k - 2\omega\, Mk\,(\vec{k}\cdot\vec{v}_o) + Mk\,(\vec{k}\cdot\vec{v}_o)^2$$

$$+ N\Omega_3^{\,2} - 2\omega\, N\,(\vec{k}\cdot\vec{v}_o) + N\,(\vec{k}\cdot\vec{v}_o)^2$$

$$+ P\Omega_1^{\,2}k - P\,c^2k^3 + Q\Omega_1^{\,2} - Q\,c^2k^2 = 1$$

Solution for the Coefficients

The preceding equation holds for all values of k.

For $k = 0$

$$Q\Omega_1^{\,2} = 1 - N\Omega_3^{\,2} \tag{E1}$$

For $k \neq 0$, we equate coefficients, writing

$$\vec{K} \cdot \vec{v}_o = |\vec{k}|\,|\vec{v}_o| \cos \lambda \ .$$

Since all values of $\vec{K}$ can be reached with $k > 0$ and the appropriate polar angles, we consider only positive values of k. Equating coefficients of like powers of k gives the following:

$$k: \quad M\Omega_3^{\,2} - 2\omega N v_o \cos \lambda + P\Omega_1^{\,2} = 0 \tag{E2}$$

$$k^2: \quad - 2\omega M v_o \cos \lambda + N v_o^{\,2} \cos^2 \lambda - Q c^2 = 0 \tag{E3}$$

$$k^3: \quad M v_o^{\,2} \cos^2 \lambda - P c^2 = 0 \ . \tag{E4}$$

Equation (E4) yields

$$P = \left(\frac{v_o}{c}\right)^2 M \cos^2 \lambda . \tag{E5}$$

We must now solve (E2) and (E3) for M and N. Using (E1) and (E5) in (E2) and (E3), we obtain

$$[\Omega_3^{\,2} + \left(\frac{v_o}{c}\right)^2 \Omega_1^{\,2} \cos^2 \lambda] M - 2\omega N v_o \cos \lambda = 0,$$

$$- [2\omega \Omega_1^{\,2} v_o \cos \lambda] M + [\Omega_1^{\,2} v_o^{\,2} \cos^2 \lambda + \Omega_3^{\,2} c^2] N = c^2.$$

We solve these equations simultaneously for M and N. From the first equation

$$M = \frac{(2\omega v_o \cos \lambda) N}{[\Omega_3^{\,2} + \left(\frac{v_o}{c}\right)^2 \Omega_1^{\,2} \cos^2 \lambda]} .$$

Substitution in the second gives

$$N \left\{ [\Omega_3{}^2 + (\frac{v_o}{c})^2 \ \Omega_1{}^2 \cos^2 \lambda]^2 - [4\omega^2 \ \Omega_1{}^2 \ (\frac{v_o}{c})^2 \ \cos^2 \lambda] \right\} =$$

$$[\Omega_3{}^2 + (\frac{v_o}{c})^2 \ \Omega_1{}^2 \cos^2 \lambda].$$

Therefore, the coefficients M and N are

$$N = \frac{[\Omega_3{}^2 + (\frac{v_o}{c})^2 \ \Omega_1{}^2 \cos^2 \lambda]}{\Omega_3{}^4 \ D}$$

$$M = \frac{2\omega \ v_o \cos \lambda}{\Omega_3{}^4 \ D} ,$$

where the denominator D is

$$D = \left\{ 1 + 2 \ (\frac{v_o}{c})^2 \ \frac{\Omega_1{}^2}{\Omega_3{}^2} \cos^2 \lambda - 4 \ (\frac{v_o}{c})^2 \ \frac{\omega^2 \ \Omega_1{}^2}{\Omega_3{}^4} \cos^2 \lambda \right.$$

$$\left. + (\frac{v_o}{c})^4 \ \frac{\Omega_1{}^4}{\Omega_3{}^4} \cos^4 \lambda \right\}.$$

Equations (E1) and (E5) supply the remaining coefficients,

$$P = \left(\frac{v_o}{c}\right)^2 M \cos^2 \lambda$$

$$Q = \frac{1}{\Omega_1^2} \left[1 - N\,\Omega_3^2\right] ,$$

in terms of M and N.*

---

* It can be shown that the expressions above for N and Q hold for <u>all</u> values of k (both $k > 0$ and $k < 0$), whereas the expressions for M and P change sign for $k < 0$. However, we note that M and P are multiplicative with k in the numerators of the partial fractions. Hence for <u>all</u> k

$$Mk = \frac{2\omega}{D\,\Omega_3^4} (\vec{k}\cdot\vec{v}_o)$$

$$Pk = \left(\frac{v_o}{c}\right)^2 \left[\frac{2\omega}{D\,\Omega_3^4} (\vec{k}\cdot\vec{v}_o)\right] \cos^2 \lambda$$

APPENDIX F

Contour Integration of $\vec{E}_{\perp}^{(1)}(\vec{x},t)$ and $\vec{E}_{\parallel}^{(1)}(\vec{x},t)$: The Inclusion of Collision Damping

For $\frac{v_o}{c} << 1$, the inverse Fourier transforms of $\vec{E}_{\perp}^{(1)}(\vec{k},\omega)$ and $\vec{E}_{\parallel}^{(1)}(\vec{k},\omega)$ are

$$\vec{E}_{\perp}^{(1)}(\vec{x},t) \simeq \frac{i}{4\pi^2 c} \nabla \times \vec{\mu}_o \int d\omega \int d\vec{k} \left\{ \frac{\omega\,[\delta(\omega-\omega_o) + \delta(\omega+\omega_o)]\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}}{(k+a)\,(k-a)} \right\} \tag{15}$$

$$\vec{E}_{\parallel}^{(1)}(\vec{x},t) \simeq \frac{\sum_i \omega_{p_i}^2}{4\pi^2} \nabla \left\{ \frac{\vec{v}_o}{c} \cdot \nabla \times \vec{\mu}_o \int d\omega \int d\vec{k} \, \frac{[\delta(\omega-\omega_o) + \delta(\omega+\omega_o)]\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}}{\left(\omega^2 - \sum_i \omega_{p_i}^2\right)(k+a)\,(k-a)} \right\}, \tag{16}$$

where $a^2 = \frac{\omega^2}{c^2}\left[1 - \frac{\sum_i \omega_{p_i}^2}{\omega^2}\right]$. These are equations (15) and (16) of the text. We must integrate over $\vec{k}$ and ω.

F.1 $\vec{k}$ Integration

The $\vec{k}$ integrands for $\vec{E}_{\perp}^{(1)}(\vec{x},t)$ and $\vec{E}_{\parallel}^{(1)}(\vec{x},t)$ are the same, viz.

$$\frac{e^{i(\vec{k}\cdot\vec{x})}}{(k+a)\,(k-a)} .$$

We perform the k space integration in spherical coordinates, letting

$$\int d\vec{k}\, \frac{e^{i(\vec{k}\cdot\vec{x})}}{(k+a)\,(k-a)} = \int_0^{2\pi} d\varphi_k \int_{-1}^{+1} d(\cos\theta_k) \int_0^{\infty} dk\, \frac{k^2\, e^{i(kx\cos\theta_k)}}{(k+a)\,(k-a)} .$$

Since the integrand is azimuthally symmetric, integration over $d\varphi_k$ yields a multiplicative factor 2π. For the θ_k integration

$$2\pi \int_{-1}^{+1} d(\cos\theta_k) \int_0^{\infty} dk \left\{ \frac{k^2\, e^{i(kx\cos\theta_k)}}{(k+a)\,(k-a)} \right\} =$$

$$\frac{2\pi}{ix} \int_0^{+\infty} dk \left\{ \frac{k\,[e^{ikx} - e^{-ikx}]}{(k+a)\,(k-a)} \right\}$$

The k integrand is now an even function of k; hence,

$$\frac{2\pi}{ix}\int_{0}^{\infty} dk \left\{ \frac{k[e^{ikx} - e^{-ikx}]}{(k+a)\ (k-a)} \right\} = \frac{\pi}{ix}\int_{-\infty}^{+\infty} dk \left\{ \frac{k[e^{ikx} - e^{-ikx}]}{(k+a)\ (k+a)} \right\}$$

$$= I\ (|\vec{x}|,\ \omega)\ ,$$

where x has been written explicitly as $|\vec{x}|$.

We use complex contour integration (Figure 5) and the residue theorem to evaluate $I(|x|,\omega)$. Some care is required since the poles of the integrand lie at $k = \pm a$ on the real axis. The integral can be made well defined in one of two ways: (i) the requirement can be imposed that only outgoing waves be present in the result; or, (ii) a small collisional damping term $-\nu_j \vec{v}_j^{(1)}$ can be added to the right hand side of Eq. (2b), where ν_j is the (assumed constant) collision frequency for the jth species.

We choose the latter procedure. Which we use makes little difference, since we usually deal with the case ω_o, $\omega_{p_j} >>$ all ν_j, so the collision frequencies do not appear in the eventual answers. But there is one important exception to this: the case all $\omega_{p_j} >>$ all $\nu_j >> \omega_o$, the "quasi d.c. - case."

When either $\omega_o > \omega_{p_j} >> \text{all } \nu_j$ or $\omega_{p_j} > \omega_o >> \text{all } \nu_j$ $\left(\frac{v_o}{c} << 1\right)$, the net effect of adding the term $-\nu_j \vec{v}_j^{(1)}$ on the right of Eq. (2b) is simply to replace the mass m_j in all the equations with $m_j\,(1 + i\nu_j/\omega)$. This means that

$$\omega_{p_j}^2 \rightarrow \frac{\omega_{p_j}^2}{\left(1 + \frac{i\nu_j}{\omega}\right)}$$

$$a^2 \rightarrow \frac{1}{c^2}\left[\omega^2 - \sum_j \frac{\omega_{p_j}^2}{\left(1 + \frac{i\nu_j}{\omega}\right)}\right] .$$

We re-write the latter expression as

$$a^2 = \frac{1}{c^2}\left\{\omega^2\left[1 - \sum_j \frac{\omega_{p_j}^2}{(\omega^2 + \nu_j^2)}\right] + i\omega\left[\sum_j \frac{\omega_{p_j}^2\ \nu_j}{(\omega^2 + \nu_j^2)}\right]\right\} .$$

and set

$$\alpha \equiv \frac{\omega^2}{c^2}\left[1 - \sum_j \frac{\omega_{p_j}^2}{(\omega^2 + \nu_j^2)}\right]$$

$$\beta \equiv \frac{\omega}{c^2}\left[\sum_j \frac{\omega_{p_j}^2}{(\omega^2 + \nu_j^2)}\ \nu_j\right] .$$

Case I: $|\omega| > \left(\sum_j \omega_{p_j}^2\right)^{\frac{1}{2}} >> \text{all } \nu_j$

More precisely, we consider the $\vec{k}$ integration when α is positive, i.e., when

$$0 < \sum_j \frac{\omega_{p_j}^2}{(\omega^2 + \nu_j^2)} < 1 .$$

Let the (now) complex $a^2 = \alpha + i\beta = \rho e^{i\theta}$, where $\rho = [\alpha^2 + \beta^2]^{1/2}$ and $\tan\theta = \frac{\beta}{\alpha}$; then $a = |\sqrt{\rho}|\, e^{i\frac{\theta}{2}}$. But the latter expression is equivalent to

$$a = |\sqrt{\rho}| \left\{ \cos\left[\frac{1}{2}\tan^{-1}\left(\frac{\beta}{\alpha}\right)\right] + i \sin\left[\frac{1}{2}\tan^{-1}\left(\frac{\beta}{\alpha}\right)\right] \right\}$$

For $\frac{\beta}{\alpha} << 1$ (with α positive), $\frac{1}{2}\tan^{-1}\left(\frac{\beta}{\alpha}\right) \simeq \frac{1}{2}\left(\frac{\beta}{\alpha}\right)$. Therefore,

$$\cos\left[\frac{1}{2}\tan^{-1}\left(\frac{\beta}{\alpha}\right)\right] \simeq \frac{2\alpha}{[4\alpha^2 + \beta^2]^{1/2}} \simeq 1$$

$$\sin\left[\frac{1}{2}\tan^{-1}\left(\frac{\beta}{\alpha}\right)\right] \simeq \frac{\beta}{[4\alpha^2 + \beta^2]^{1/2}} \simeq \frac{\beta}{2\alpha}$$

and

$$a \simeq \left|\sqrt{\rho}\right| \left\{ 1 + i\,\frac{\beta}{2\alpha} \right\}.$$

For $\omega > 0$, β is positive, whereas for $\omega < 0$, β is negative. Hence, the complex poles are slightly above or slightly below the real axis, depending on the sign of ω (Figure 5A). By Cauchy's theorem, for $\omega > 0$.

$$\int_{-\infty}^{+\infty} dk \quad \frac{k e^{ikx}}{(k+a)\ (k-a)} \quad \xrightarrow[\text{contour}]{\text{upper}} \simeq \pi i e^{i(\sqrt{\rho})|\vec{x}|}\ e^{-(\sqrt{\rho})\ \left(\frac{\beta}{2\alpha}\right)\ |\vec{x}|}$$

$$\int_{-\infty}^{+\infty} dk \quad \frac{k e^{-ikx}}{(k+a)\ (k-a)} \quad \xrightarrow[\text{contour}]{\text{lower}} \simeq -\pi i e^{i(\sqrt{\rho})|\vec{x}|}\ e^{-(\sqrt{\rho})\ \left(\frac{\beta}{2\alpha}\right)\ |\vec{x}|}$$

Therefore,

$$I_{\omega > \omega_p}(|\vec{x}|,\ \omega) = \frac{2\pi^2}{|\vec{x}|} \quad e^{i(\sqrt{\rho})\ |\vec{x}|} \quad e^{-(\sqrt{\rho})\ \left(\frac{\beta}{2\alpha}\right)\ |\vec{x}|}$$

Similarly, when $\omega < 0$

$$I_{-\omega < -\omega_p}(|\vec{x}|,\omega) = \frac{2\pi^2}{|\vec{x}|} e^{-i(\sqrt{\rho})|\vec{x}|} e^{-(\sqrt{\rho})(\frac{\beta}{2\alpha})|\vec{x}|}$$

<u>Case II: $\left(\sum_i \omega_{p_j}^2\right)^{\frac{1}{2}} > |\omega| >> \text{all } \nu_j$</u>

Again, to be more precise, we consider a case where α is negative, i.e., where

$$\sum_j \frac{\omega_{p_j}^2}{(\omega^2 + \nu_j^2)} > 1;$$

but we still assume that $\left|\frac{\beta}{\alpha}\right| << 1$. Hence, $\tan^{-1}(\frac{\beta}{\alpha})$ is slightly less than or slightly greater than 180°, respectively, depending on whether ω (and therefore β) is greater than or less than zero. As a result

$$\cos\left[\frac{1}{2}\tan^{-1}\left(\frac{\beta}{\alpha}\right)\right] \simeq \frac{\frac{\beta}{2}}{\left|\left[\alpha^2 + \frac{\beta^2}{4}\right]^{\frac{1}{2}}\right|} \simeq \frac{\beta}{2|\alpha|}$$

$$\sin\left[\frac{1}{2}\tan^{-1}\left(\frac{\beta}{\alpha}\right)\right] \simeq \frac{|\alpha|}{\left|\left[\alpha^2 + \frac{\beta^2}{4}\right]^{\frac{1}{2}}\right|} \simeq 1,$$

and

$$a \simeq |\sqrt{\rho}| \left\{ \frac{\beta}{2|\alpha|} + i \right\}$$

For ω (and β) > 0, the pole is located just to the right of the positive imaginary axis at a vertical distance α from the origin, whereas for ω (and β) < 0, the pole is located just to the left of the positive imaginary axis at the same distance from the origin (Figure 5B). Using Cauchy's theorem, we again calculate $I\ (|\vec{x}|,\omega)$. For $\omega > 0$, the two terms of $I\ (|\vec{x}|,\omega)$ are

$$\int_{-\infty}^{+\infty} dk \ \frac{ke^{ikx}}{(k+a)\ (k-a)} \ \xrightarrow[\text{contour}]{\text{upper}} \ \simeq \ \pi i e^{-(\sqrt{\rho})|\vec{x}|} \ e^{i(\sqrt{\rho})\ \left|\frac{\beta}{2\alpha}\right| |\vec{x}|}$$

$$\int_{-\infty}^{+\infty} dk \ \frac{ke^{-ikx}}{(k+a)\ (k-a)} \ \xrightarrow[\text{contour}]{\text{lower}} \ \simeq \ -\pi i e^{-(\sqrt{\rho})|\vec{x}|} \ e^{i(\sqrt{\rho})\ \left|\frac{\beta}{2\alpha}\right| |\vec{x}|}$$

so that

$$I_{\omega < \omega_p} = \frac{2\pi^2}{|\vec{x}|} \ e^{-(\sqrt{\rho})|\vec{x}|} \ e^{i(\sqrt{\rho})\ \left|\frac{\beta}{2\alpha}\right| |\vec{x}|} \ .$$

It can be shown that for $\omega < 0$

$$I_{-\omega_p < -\omega} = \frac{2\pi^2}{|\vec{x}|} \; e^{-(\sqrt{\rho})|\vec{x}|} \; e^{-i(\sqrt{\rho}) \left|\frac{\beta}{2\alpha}\right| |\vec{x}|}$$

<u>Case III $\left(\sum_j \omega_{p_j}^2\right)^{\frac{1}{2}} >> \text{all } \nu_j > |\omega|$</u>

As $\vec{v}_o$ goes to zero and as ω becomes vanishingly small (less than the ν_j), the poles lie approximately on the 45° diagonals in the complex k plane. Therefore,

$$I_{\omega < \nu < \omega_p} = \frac{2\pi^2}{|\vec{x}|} \; e^{-\left(\sqrt{\frac{\rho}{2}}\right)|\vec{x}|} \; e^{i\left(\sqrt{\frac{\rho}{2}}\right)|\vec{x}|}$$

$$I_{-\omega_p < -\nu < -\omega} = \frac{2\pi^2}{|\vec{x}|} \; e^{-\left(\sqrt{\frac{\rho}{2}}\right)|\vec{x}|} \; e^{-i\left(\sqrt{\frac{\rho}{2}}\right)|\vec{x}|}$$

<u>F.2. ω Integration</u>

The ω integration is facilitated by the presence of delta functions.

<u>Case I: $|\omega_o| > \left(\sum_i \omega_{p_i}^2\right)^{\frac{1}{2}} >> \text{all } \nu_j$</u>

We break the ω integral up into frequency intervals. In equation (15) for $\vec{E}_\perp^{(1)}(\vec{x},t)$, we have

$$\int_{0}^{+\infty} d\omega \; [\delta(\omega - \omega_o)] \; \omega e^{-i\omega t} \; [I_{\omega > \omega_p}(|\vec{x}|, \omega)]$$

$$+ \int_{-\infty}^{0} d\omega \, [\delta(\omega + \omega_o)] \; \omega e^{-i\omega t} \, [I_{-\omega < -\omega_p} \; (|x|, \omega)] =$$

$$\frac{4\pi^2 i}{|\vec{x}|} \quad \omega_o e^{-(\sqrt{\rho})\left|\frac{\beta}{2\alpha}\right| |\vec{x}|} \left\{ \sin\,[(\sqrt{\rho})|\vec{x}| - \omega_o t] \right\} \tag{F2}$$

On the other hand, in equation (16) for $\vec{E}_{\parallel}^{(1)}(x,t)$, we must be careful to avoid integrating over frequencies within $\pm\, \epsilon$ of $\left(\sum_i \omega_{p_i}^{2}\right)^{\frac{1}{2}}$ (see Case IV). For the frequency range $|\omega_o| > \left(\sum_i \omega_{p_i}^{2}\right)^{\frac{1}{2}} + |\epsilon|$, the ω integration yields

$$\frac{4\pi^2}{|\vec{x}|} \; \frac{\left(\sum_i \omega_{p_i}^{2}\right) \; e^{-(\sqrt{\rho})\left|\frac{\beta}{2\alpha}\right| \; |\vec{x}|}}{\left[\omega_o^{2} - \left(\sum_i \omega_{p_i}^{2}\right)\right]} \quad \cos\,[(\sqrt{\rho}) \; |\vec{x}| - \omega_o t] \; . \tag{F3}$$

Case II: $\left(\sum_j \omega_{p_j}^{\;2}\right)^{\frac{1}{2}} > |\omega_o| >> \text{all } \nu_j$

Integration over ω gives

$$\frac{4\pi^2 i}{|\vec{x}|}\,\omega_o\, e^{-(\sqrt{\rho})|\vec{x}|}\left\{\sin\left[(\sqrt{\rho})\left|\frac{\beta}{2\alpha}\right|\,|\vec{x}|\;-\omega_o t\right]\right\} \tag{F4}$$

in the expression for $\vec{E}_\perp^{(1)}$, while for $\vec{E}_\parallel^{(1)}$ we have

$$-\frac{4\pi^2}{|\vec{x}|}\;\frac{\left(\sum_i \omega_{p_i}^{\;2}\right)\; e^{-(\sqrt{\rho})|\vec{x}|}}{\left(\sum_i \omega_{p_i}^{\;2} - \omega_o^{\;2}\right)}\;\cos\left[(\sqrt{\rho})\left|\frac{\beta}{2\alpha}\right||\vec{x}| - \omega_o t\right]. \tag{F5}$$

We set the "damping decrement," $\sqrt{\rho}$, in these expressions equal to b:

$$b = \left\{\frac{\omega_o^{\;4}}{c^4}\left[\sum_j \frac{\omega_{p_j}^{\;2}}{(\omega_o^{\;2}+\nu_j^{\;2})} - 1\right]^2 + \frac{\omega_o^{\;2}}{c^4}\left[\sum_j \frac{\omega_{p_j}^{\;2}\;\nu_j}{(\omega_o^{\;2}+\nu_j^{\;2})}\right]^2\right\}^{\frac{1}{4}}$$

<u>Case III: $\left(\sum_j \omega_{p_j}^2\right)^{\frac{1}{2}} >> \text{all } \nu_j > |\omega_o|$</u>

We have

$$\frac{4\pi^2 i}{|\vec{x}|} \quad \omega_o \, e^{-\frac{b}{\sqrt{2}}|\vec{x}|} \sin\left[\frac{b}{\sqrt{2}}|\vec{x}| - \omega_o t\right]$$

after the ω integration in (15). Expression (16) vanishes when $\vec{v}_o$ goes to zero.

<u>Case IV: $\left(\sum_i \omega_{p_i}^2\right)^{\frac{1}{2}} - \epsilon < |\omega| < \left(\sum_i \omega_{p_i}^2\right)^{\frac{1}{2}} + \epsilon$</u>

In performing the ω integration to obtain (F3) and (F5), we did not include collisional effects. That is, we did not substitute $m_j\,(1 + i\nu_j/\omega)$ for m_j in the ω_{p_j} which appear <u>outside</u> $I(|\vec{x}|\,,\,\omega)$. The reason is that we shall presently consider these expressions in their collisionless limit, having used the collision frequencies merely to facilitate the k integration. Strictly speaking, however, the preservation of an infinitesimally small collision frequency may be necessary mathematically in order to successfully integrate over the frequency range defined above.

The problem here is that our expansion of the denominator D in Appendix E is invalid. Therefore, for those frequencies only, we keep D intact and attempt to integrate over ω first instead of over $\vec{k}$. Our ω integrands will be of the form

$$\int d\omega \; \frac{\delta(\omega \pm \omega_o)}{\left(\omega^2 - \sum_i \omega_{p_i}^2\right)} \quad \text{times } f\,(\vec{k},\omega) \;,$$

where $f(\vec{k},\omega)$ is some function of $\vec{k}$ and ω. We note immediately that such an integrand is undefined, i.e., 0/0, for frequencies such that $\omega^2 = \sum_i \omega_{p_i}^2$, since by definition the delta function is zero whenever $\omega \neq \omega_o$ and since we have excluded the driving frequencies from this interval.

Inserting the collision frequencies into the integrand through the ω_{p_j} makes the denominator complex so that it does not vanish as ω passes through $\left(\sum_i \omega_{p_i}^2\right)^{\frac{1}{2}}$. Hence, we may argue that integration over frequencies near the plasma frequency should not affect our results, provided ω_o is far enough removed from $\left(\sum_i \omega_{p_i}^2\right)^{\frac{1}{2}}$.

F.3. The Collisionless Limit

Case I: $|\omega_o| > \left(\sum_j \omega_{p_j}^{2}\right)^{\frac{1}{2}} >> \text{all } \nu_j$

In the limit of a collisionless plasma, as $\nu_j \to 0$ (or $\beta \to 0$), expressions (F2) and (F3) become

$$\frac{4\pi^2 i}{|\vec{x}|} \quad \omega_o \quad \sin\,(a\,|\vec{x}| - \omega_o t)$$

and

$$\frac{4\pi^2}{|\vec{x}|} \quad \frac{\left(\sum_i \omega_{p_i}^{2}\right)}{\left(\omega_o^{2} - \sum_i \omega_{p_i}^{2}\right)} \quad \cos \quad (a\,|\vec{x}| - \omega_o t) \quad ,$$

respectively, where

$$a = \frac{\omega_o}{c}\left[1 - \frac{\sum_i \omega_{p_i}^{2}}{\omega_o^{2}}\right]^{\frac{1}{2}} \quad .$$

We substitute these results into equations (15) and (16) and obtain equations (17) and (18) of the text:

$$\vec{E}_{\perp}^{(1)}(\vec{x},t) = -\frac{\omega_o}{c}\ \nabla \times \left[\frac{\vec{\mu}_o \sin\left(a|\vec{x}| - \omega_o t\right)}{|\vec{x}|}\right] \tag{17}$$

$$\vec{E}_{\parallel}^{(1)}(\vec{x},t) = \frac{\left(\sum_i \omega_{p_i}^{\ 2}\right)}{\left(\omega_o^{\ 2} - \sum_i \omega_{p_i}^{\ 2}\right)}\ \nabla\left\{\frac{\vec{v}_o}{c} \cdot \nabla \times \vec{\mu}_o \frac{\cos(a|\vec{x}| - \omega_o t)}{|\vec{x}|}\right\} \tag{18}$$

<u>Case II: $\left(\sum_i \omega_{p_i}^{\ 2}\right)^{\frac{1}{2}} > |\omega_o| >> \text{all } \nu_j$</u>

In the collisionless limit, but with the explicit restriction that $|\omega_o| >>$ all the ν_j, expressions (F4) and (F5) go over into

$$-\frac{4\pi^2 i}{|\vec{x}|}\ \omega_o\ e^{-b|\vec{x}|} \sin \omega_o t$$

and

$$- \frac{4\pi^2}{|\vec{x}|} \frac{\left(\sum_i \omega_{p_i}^2\right) e^{-b|\vec{x}|}}{\left(\sum_i \omega_{p_i}^2 - \omega_o^2\right)} \cos \omega_o t ,$$

respectively, where

$$b = \frac{\left(\sum_i \omega_{p_i}^2\right)^{\frac{1}{2}}}{c} \left[1 - \frac{\omega_o^2}{\sum_i \omega_{p_i}^2}\right]^{\frac{1}{2}} .$$

We substitute these results into equations (15) and (16) and recover equations (19) and (20) of the text.:

$$\vec{E}_{\perp}(\vec{x},t) = \frac{\omega_o}{c} \nabla \times \left\{ \frac{\vec{\mu}_o \; e^{-b|\vec{x}|}}{|\vec{x}|} \sin \omega_o t \right\} \tag{19}$$

$$\vec{E}_{\parallel}(\vec{x},t) = - \frac{\left(\sum_i \omega_{p_i}^2\right) \cos \omega_o t}{\left(\sum_i \omega_{p_i}^2 - \omega_o^2\right)} \nabla \left\{ \frac{\vec{v}_o}{c} \cdot \nabla \times \vec{\mu}_o \frac{e^{-b|\vec{x}|}}{|\vec{x}|} \right\} . \tag{20}$$

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FIGURE CAPTIONS

Figure 1. A circular loop of source current immersed in a streaming plasma. The zeroth order plasma flow vector is $\vec{v}_o$. The unit vectors $\hat{e}_x$, $\hat{e}_y$ and $\hat{e}_z$ form a Cartesian triad with $\hat{e}_z$ normal to the plane of the loop; r and φ are the corresponding polar coordinates. The radius of the loop is a_o.

Figure 2. Coordinate configuration for the electric and magnetic fields. The Cartesian coordinate system (xyz) is fixed in space with its y axis directed along $\vec{v}_o$; the xz plane is normal to the direction of plasma flow. The vector $\vec{\mu}_o$ represents the magnetic moment of the point dipole, which is placed at the origin of (xyz). The polar angle θ_m and the azimuthal angle φ_m specify the angular orientation of $\vec{\mu}_o$, while the spherical coordinates r, θ, and φ define the position of an observer.

Figure 3. Poynting vector skew. The dipole is perpendicular to the plane of the paper (the xy plane of Figure 2), while the plasma flow is parallel to it. The dark arrows show schematically the direction of the Poynting vector, $\vec{S}$, in the xy plane at various points around a contour of constant $E_\perp^2$. The

angle τ, which specifies the degree of skew, or tilt, of $\vec{S}$ away from the radial direction, is defined by the expression

$$\tan \tau = \frac{|\vec{v}_o|}{c} \left[\frac{\left(\sum_i \omega_{p_i}^2\right)}{\omega_o^2 \; n_a} \right] |\cos \varphi| \, .$$

Figure 4. Contours of constant E^2 and $E_{\parallel}^2$ (radiation fields). The dipole is perpendicular to the zeroth-order plasma flow, as in Figure 3. The ellipse schematically represents a contour of constant E^2, where $|\vec{E}|$ is the magnitude of the total electric field. The lemniscate schematically traces a contour of constant $E_{\parallel}^2$. The contours lie in the "equatorial" (xy) plane of the dipole. The time-average force which the plasma exerts on the dipole is

$$\vec{F}_{orce} = \left[\frac{\mu_o^2 \; \omega_o^2 \left(\sum_i \omega_{p_i}^2\right) n_a}{6 \, c^4} \right] \frac{\vec{v}_o}{c}$$

Figures 5A and 5B Complex integration contours in $\vec{k}$ space (Appendix F).

Figure (5A) applies to the case where

$$|\omega_o| > \left(\sum_i \omega_{p_i}^2\right)^{\frac{1}{2}} >> \text{all } \nu_j \, ,$$

while Figure (5B) is applicable to

$$\left(\sum_i \omega_{p_i}^2\right)^{\frac{1}{2}} > |\omega_o| >> \text{all } \nu_j \quad .$$

The encircled x's are poles of the k integrands.

G67-382

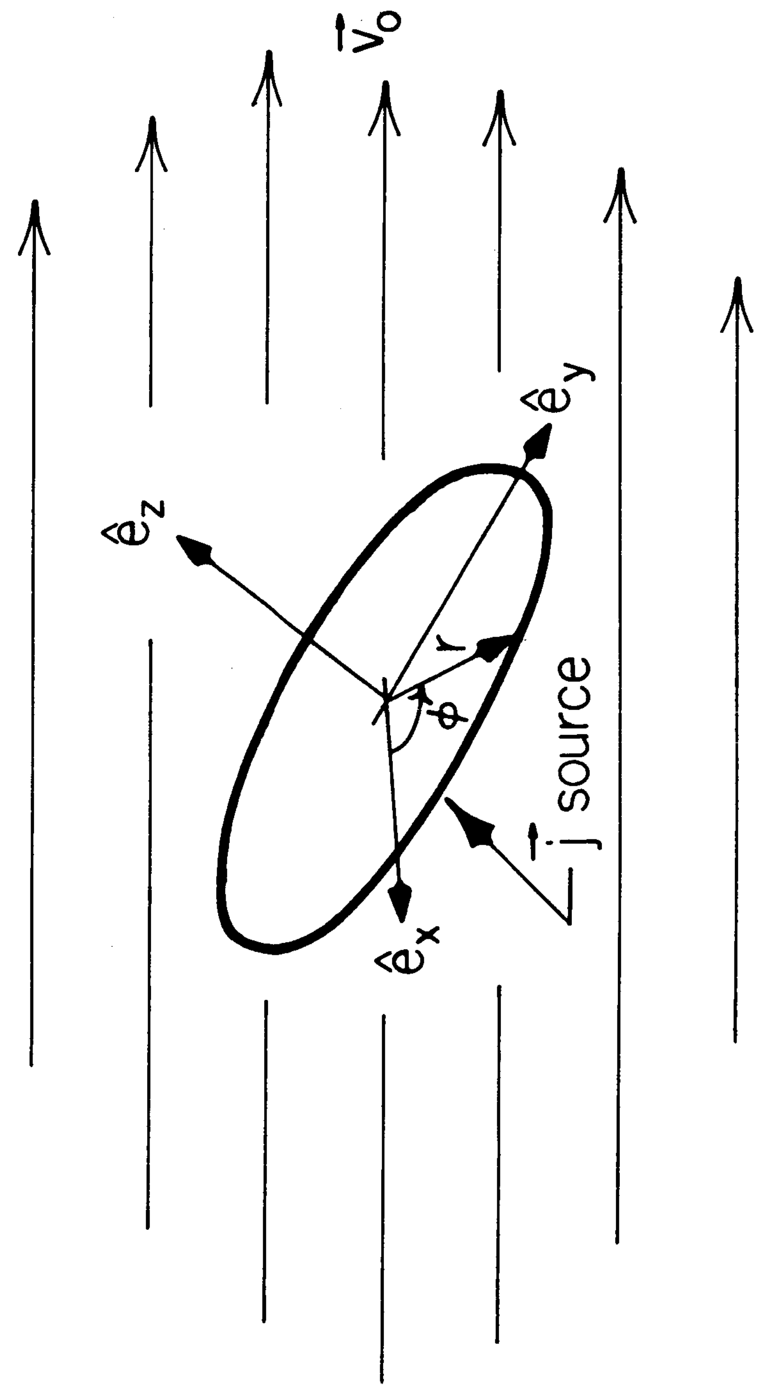

FIGURE 1

G 67-383

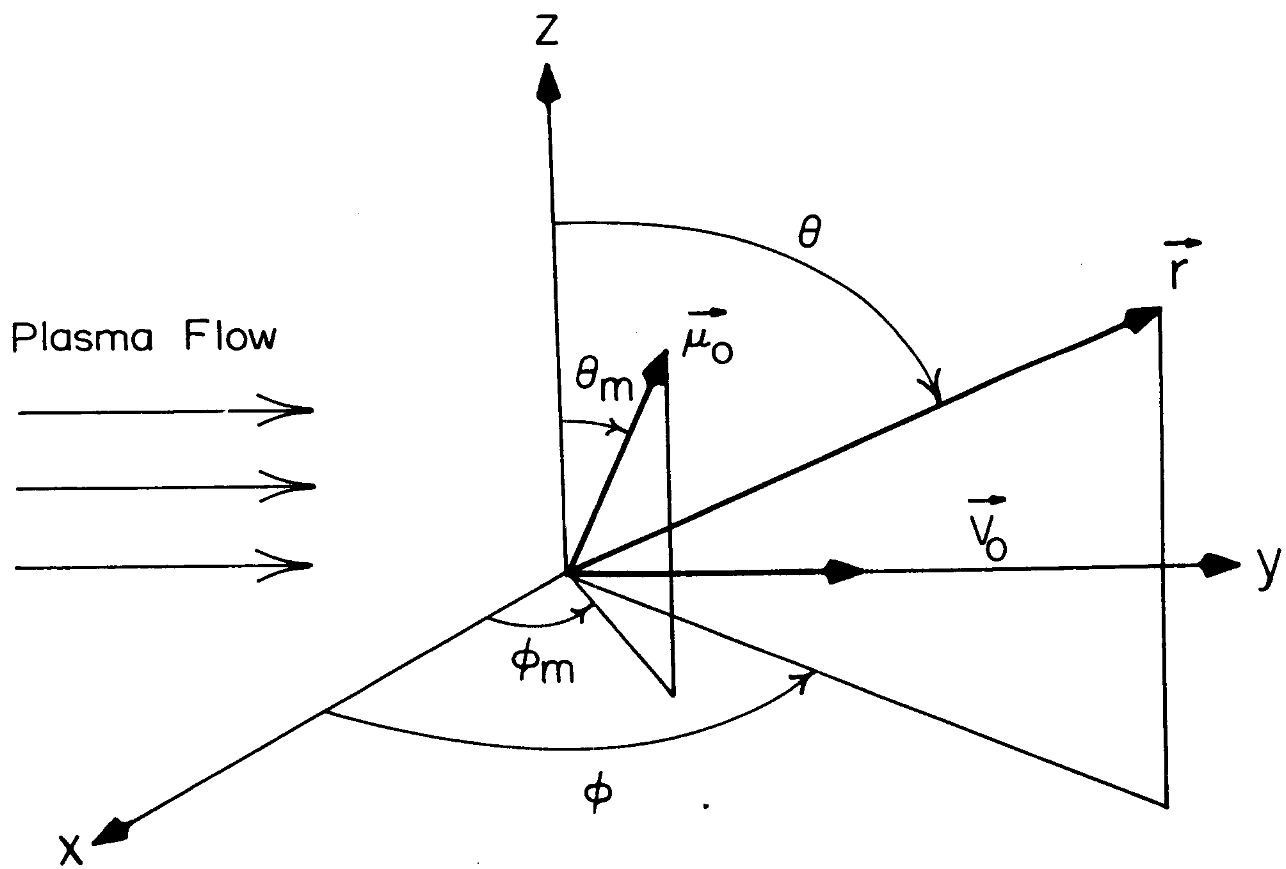

FIGURE 2

G67-384

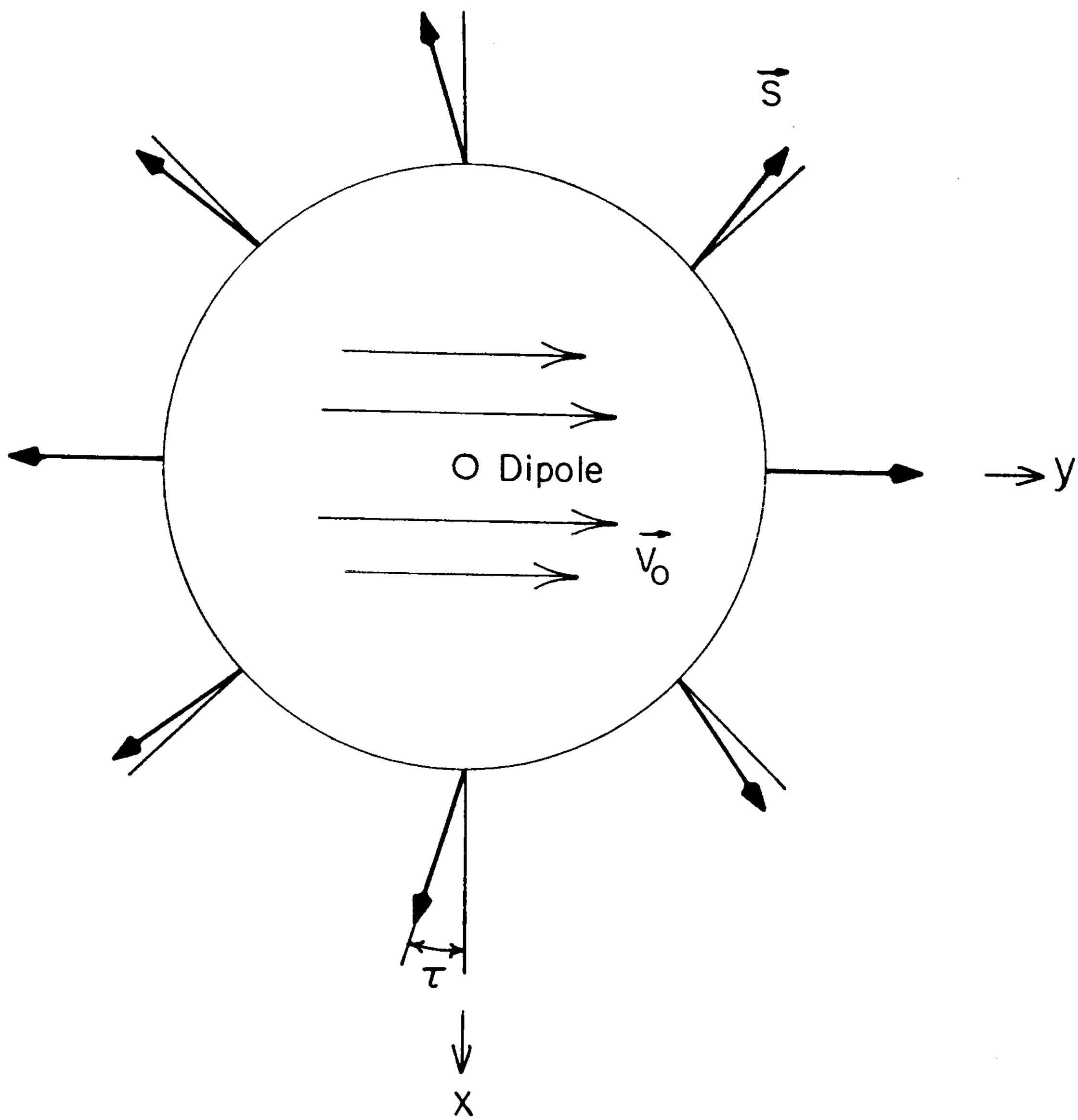

FIGURE 3

G67-385

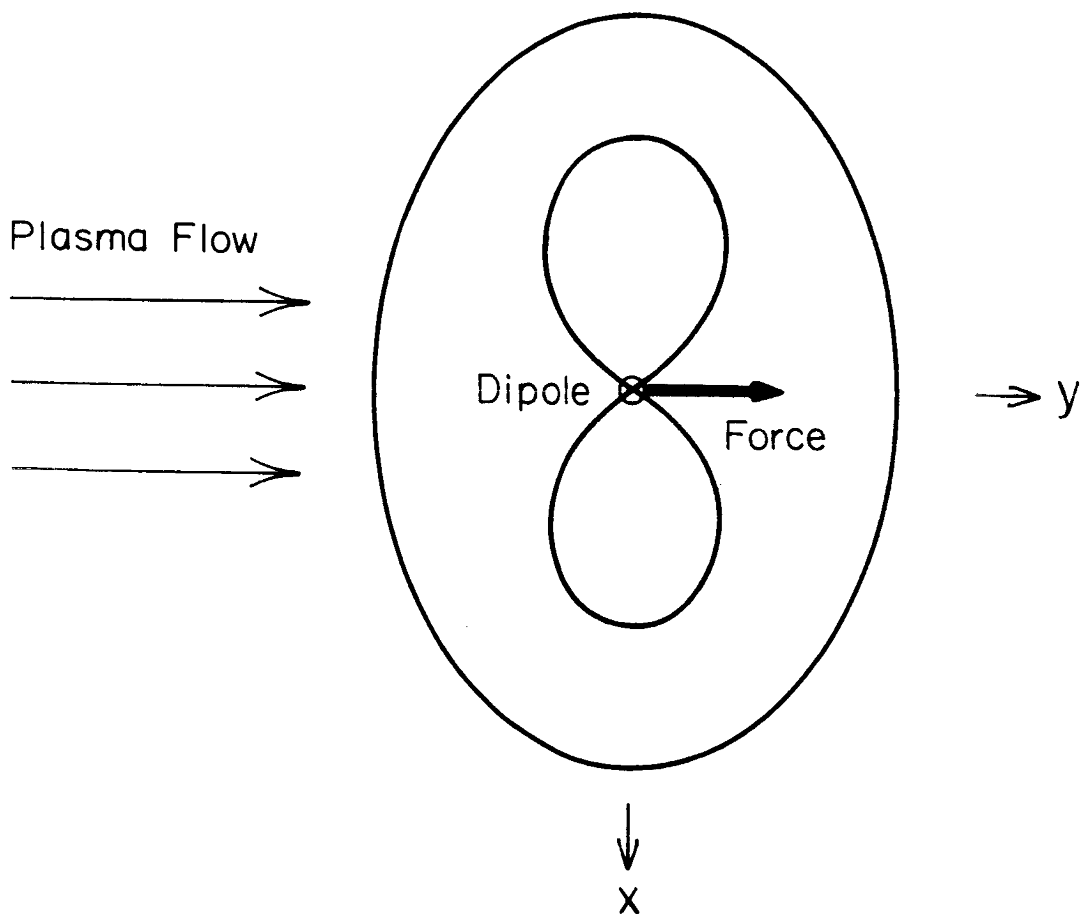

FIGURE 4

G67-386

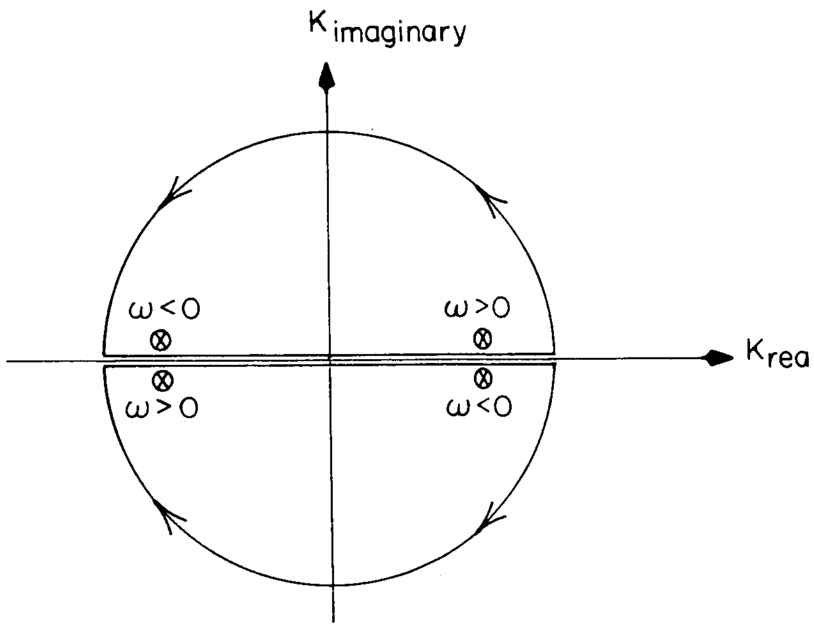

FIGURE 5A

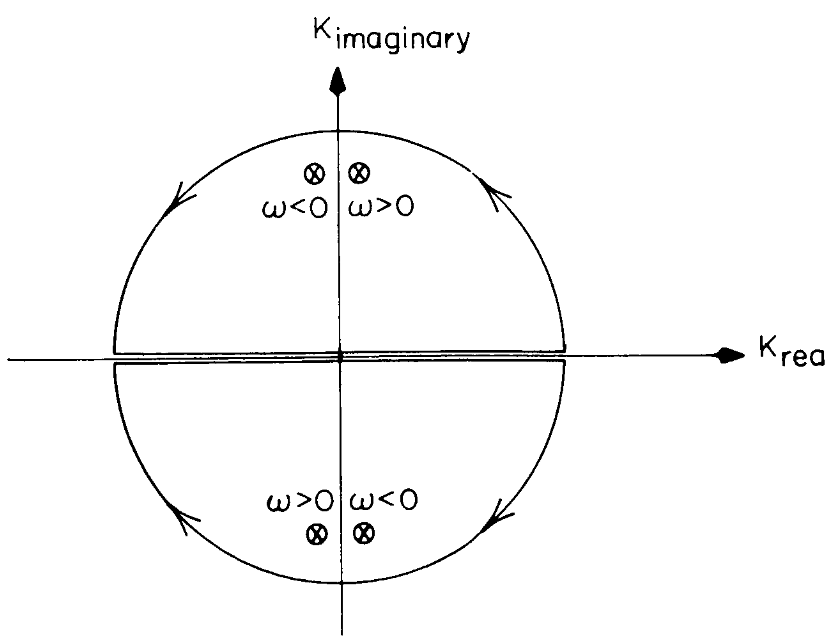

FIGURE 5B